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Baryon masses in a nonrelativistic model with the quantum isotonic oscillator potential

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The nonrelativistic quark model and a new baryon mass formula have been applied to study the baryon octet and decuplet masses. To describe the quark-quark interacting forces inside baryons, a suitable phenomenological form of the potential and quantum isotonic oscillator potential have been proposed. A comparison between calculations reported in this study and the available experimental data is investigated. The description of the spectrum shows that the position of the Roper resonances of the nucleon, the ground states and the excited multiplets up to three GeV are in general well reproduced.

Keywords: Schrödinger equation, Baryons spectra, Isotonic oscillator potential, Ansatz method

1 Introduction

The baryons have been made up of three constituent confined quarks and there are several attempts to calculate the baryon masses in various models[1-8]. The hypercentral Constituent Quark Models (hCQM) have been recently widely applied to the description of baryon properties and most attention has been devoted to the spectrum. The baryon spectrum is usually described well, although the various models are quite different. Common to these models is the fact that the three quark interaction can be divided in two parts. First, containing the confinement interaction is spin and flavour are independent and is therefore SU (6) invariant, while the second violates the SU (6) symmetry[1,9-12]. It is well known that the Gürsey Radicati mass formula[13] describes quite well the way SU (6) symmetry is broken, at least in the lower part of the baryon spectrum. In this work we want to apply the generalized Gürsey Radicati (GR) mass formula which is presented by Giannini *et al*[14] to calculate the baryon masses. The model we used is a simple CQM where the SU (6) invariant part of the Hamiltonian is the same as in the hypercentral constituent quark model (hCQM)[15,16] and where the SU (6) symmetry is broken by a generalized GR mass formula. The exact solution of the Schrodinger equation for the isotonic oscillator potential via wave function ansatz is given and the generalized GR mass formula is introduced. The obtained results have been analyzed by fitting the parameters of the generalized GR mass formula to the octet and decuplet baryon masses and the spectrum with the experimental data is compared.

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2 Theoretical Model

In the six-dimensional, the Schrödinger equation for a system containing three particles with a potential $V(r)$ and by considering of $\psi_{\nu\gamma} = r^{-5/2} R_{\nu\gamma}(r)$ can be written as:

$$H\psi_{\nu\gamma} = E\psi_{\nu\gamma} \rightarrow \left[\frac{d^2}{dr^2} - \frac{\eta^2 - \frac{1}{4}}{r^2} + 2m(E - V(r))\right] R_{\nu\gamma}(r) = 0 \quad \ldots (1)$$

where $\eta = \gamma + 2$. $R_{\nu\gamma}(r)$, r and γ are the hyperradial wave function, the hyperradius and the grand angular quantum number, respectively. γ is also given by $\gamma = 2n + l_\rho + l_\lambda,\ 0 \le n \le \infty$ with the angular momenta l_ρ and l_λ which are associated with the Jacobi coordinates ($\vec{\rho}$ and $\vec{\lambda}$)[17] and ν denotes the number of nodes of the space three quark wave functions. In Eq. (1) m is the reduced mass[18] which is defined as $m = \frac{2m_\rho m_\lambda}{m_\rho + m_\lambda}$. The search for exact solutions to quantum-mechanical models with rational potentials has been a very significant research aspect in the past decades. However it is well recognized that only a very limited number of models in quantum mechanics can be solved exactly. The hypercentral potentials could be of any form[19,20] (e.g., linear, log, power law, etc.) but all of these proposed potentials are not complete and perfect. In our model, the interaction potential is assumed as:

$$V(r) = \omega r^2 - 2g\frac{\left(2r^2-1\right)}{\left(2r^2+1\right)^2} \quad \dots (2)$$

Goldman and Krivchenkov demonstrated that the energy spectrum of the potential $V(r) = V_0(\frac{r}{a}-\frac{a}{r})^2, r>0$ is isomorphous to the harmonic oscillator spectrum, i.e., it consists of an infinite set of equidistant energy levels. For this reason this oscillator called "the isotonic oscillator"[21,22]. Recently, the generalized quantum isotonic oscillator Hamiltonian known by $H = -\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\omega^2 r^2+\frac{2g\left(r^2-a^2\right)}{\left(r^2+a^2\right)^2}, g>0$ is studied analytically to find eigenvalues and spectrum of this Hamiltonian by some of researches[23-26]. The interests of these problems lay on the fact that it is exactly solvable for certain values of the parameters, namely $g=2$ and $\omega a^2 = \frac{1}{2}$ where it is the case of supersymmetric problem of the harmonic oscillator[23]. Kraenkel and Senthilvelan applied the problem with a position dependent effective mass, which, with adequate mass distributions, may represent different problems encountered in semiconductor physics[24]. Sesma transformed the Schrödinger equation regarding to quantum isotonic oscillator Hamiltonian into a confluent Heun equation using a Mobius transformation and thereby obtain an efficient algorithm to solve the Schrödinger equation numerically[25]. Generalized isotonic oscillators can be seemed as possible representations of realistic quantum dots[25]. The behavior of the quantum isotonic oscillator can be seen in Fig. 1.

By substituting Eq. (2) in to Eq. (1) we obtain the following equation:

$$\left[\begin{array}{l}\frac{d^2}{dr^2}-\frac{\eta^2-\frac{1}{4}}{r^2}+2mE-2m\omega r^2 \\ +4mg\frac{\left(2r^2-1\right)}{\left(2r^2+1\right)^2}\end{array}\right]R_{\nu\gamma}(r)=0 \quad \dots (3)$$

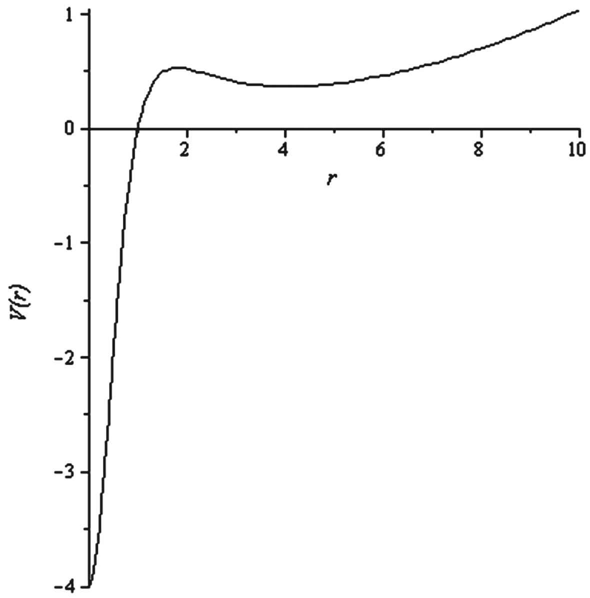

Fig. 1–$V(r)$ (in MeV) with $g=2, a=1, \omega=0.1$

We suppose the following form for the wave function

$$R_{\nu\gamma}(r) = g(r)\exp(f(r)) \quad \dots (4)$$

In the quasi-exact ansatz technique, which is in fact a special case of Lie algebraic approach, an ansatz solution is proposed based on an associated Riccati differential equation. Next, by inserting the proposed ansatz in the equation and obtaining a set of equations, the unknown coefficients in the wave function are determined. However, just like any quasi-exact approach, the technique has its limitation, namely, it imposes some restrictions on the potential parameters and that the calculation of higher-state solutions is a rather cumbersome task due to the arising set of equations. In fact, the ansatz method is useful only for the very low0lying states.

Now for the functions $f(r)$ and $g(r)$ we make use of the ansatz[27-30]:

$$g(r)=\begin{cases}1 & \nu=0\\ \prod_i^{\nu}\left(r-\alpha_i^{\nu}\right) & \nu\ge 1\end{cases} \quad \dots (5)$$

$$f(r) = -\alpha r^2+\beta Ln\, r+\lambda Ln\,(2r^2+1)$$

From Eq. (4) we obtain:

$$R''_{\nu,\gamma}(r)=\left[\begin{array}{l}f''(r)+f'^2(r)\\ +\frac{2f(r)g'(r)+g''(r)}{g(r)}\end{array}\right]R_{\nu,\gamma}(r) \quad \dots (6)$$

and from Eq. (5) we have:

$$f'(r) = -2\alpha r + \frac{\beta}{r} + \lambda\frac{4r}{2r^2+1}$$
$$f'^2(r) = 4\alpha^2 r^2 + \frac{\beta^2}{r^2} - 4\alpha\beta + \frac{16\lambda^2 r^2}{(2r^2+1)^2} - 16\alpha\lambda\frac{r^2}{2r^2+1} + 8\frac{\beta\lambda}{2r^2+1} \quad \ldots (7)$$
$$f''(r) = -2\alpha - \frac{\beta}{r^2} + \lambda\frac{4(2r^2+1)-16r^2}{(2r^2+1)^2}$$

Substitution of Eq. (7) into Eq. (6) leads to:

$$R''_{\nu,\gamma}(r) = \begin{bmatrix} -2\alpha - \frac{\beta}{r^2} + \frac{4\lambda}{(2r^2+1)^2} + 4\alpha^2 r^2 \\ -\frac{8\lambda r^2}{(2r^2+1)^2} + \frac{\beta^2}{r^2} + \frac{8\beta\lambda}{2r^2+1} \\ +\frac{16\lambda^2 r^2}{(2r^2+1)^2} - \frac{16\alpha\lambda r^2}{2r^2+1} - 4\alpha\beta \end{bmatrix} R_{\nu,\gamma}(r) \quad \ldots (8)$$

or

$$R''_{0,\gamma}(r) = \left\{ \begin{matrix} \frac{1}{r^2}(\beta^2-\beta) + \frac{1}{(2r^2+1)^2} \times \\ \begin{bmatrix} -8\lambda r^2 + 4\lambda + 16\lambda^2 r^2 + 16\beta\lambda r^2 \\ +8\beta\lambda + 16\alpha\lambda r^2 + 8\alpha\lambda \end{bmatrix} \\ -8\lambda\alpha - 2\alpha + 4\alpha^2 r^2 - 4\alpha\beta \end{matrix} \right\} R_{0,\gamma}(r) \quad \ldots (9)$$

after some simplicity. From Eq. (3), we have:

$$R''_{\nu,\gamma}(r) = \begin{pmatrix} \frac{\eta^2 - \frac{1}{4}}{r^2} - 2mE + 2m\omega r^2 \\ -4mg\frac{(2r^2-1)}{(2r^2+1)^2} \end{pmatrix} R_{\nu\gamma}(r) \quad \ldots (10)$$

By Comparing Eqs. (9) and (10), it can be found that:

$$\begin{aligned} &(\beta^2-\beta) = \eta^2 - \frac{1}{4} \\ &8\lambda\alpha + 2\alpha + 4\alpha\beta = 2mE \\ &4\alpha^2 = 2m\omega \\ &-8\lambda + 16\lambda^2 + 16\beta\lambda + 16\alpha\lambda = -8mg \\ &4\lambda + 8\beta\lambda + 8\alpha\lambda = 4mg \end{aligned} \quad \ldots (11)$$

Eq. (11) immediately yields:

$$\beta = \frac{1}{2}\left(1 + \sqrt{1 + 4(\eta^2 - \frac{1}{4})}\right)$$
$$\alpha = \sqrt{\frac{m\omega}{2}}$$
$$\gamma = \frac{mg}{2 + \sqrt{1+4(\eta^2-\frac{1}{4})} + 2\sqrt{\frac{m\omega}{2}}} \quad \ldots (12)$$

and the energy can be obtained by:

$$E = \frac{1}{m}\begin{pmatrix} 2 + \sqrt{1+4(\eta^2-\frac{1}{4})} \\ +4\frac{mg}{2+\sqrt{1+4(\eta^2-\frac{1}{4})} + 2\sqrt{\frac{m\omega}{2}}} \end{pmatrix}\sqrt{\frac{m\omega}{2}} \quad \ldots (13)$$

The hypercentral constituent quark model is fairly good for description the baryon spectrum[31], but in some cases the splitting within the various $SU(6)$ multiplets are too low. The preceding results[12,15,32] show that both spin and isospin dependent terms in the quark Hamiltonian are important. Description of the splitting within the $SU(6)$ baryon multiplets is presented by the Gürsey Radicati mass formula[13]:

$$M = M_0 + CC_2[SU_S(2)] + DC_1[U_Y(1)] + E[C_2[SU_I(2)] - \frac{1}{4}(C_1[U_Y(1)])^2] \quad \ldots (14)$$

where M_0 is the average energy value of the $SU(6)$ multiplet, $C_2[SU_S(2)]$ and $C_2[SU_I(2)]$ are the $SU(2)$ (quadratic) Casimir operators for spin and isospin, respectively, and $C_1[UY(1)]$ is the Casimir operator for the $U(1)$ subgroup generated by the hypercharge Y [33,34]. This mass formula has tested to be successful in the description of the ground state baryon masses, however, as stated by the authors themselves, it is not the most general mass formula that can be written on the basis of a broken $SU(6)$ symmetry. In order to generalize Eq. (14), Giannini *et al*[14] considered a dynamical spin-flavor symmetry $SU_{SF}(6)$ and described the $SU_{SF}(6)$ symmetry breaking mechanism by generalizing Eq. (14) as:

$$M = M_0 + AC_2[SU_{SF}(6)] + BC_2[SU_F(3)] + CC_2[SU_S(2)] + DC_1[U_Y(1)] + E[C_2[SU_I(2)] - \frac{1}{4}(C_1[U_Y(1)])^2] \quad \ldots (15)$$

In Eq. (15) the spin term $(CC_2[S_2(2)])$ represents the spin-spin interactions, the flavor term $(BC_2[SU_F(3)])$ denotes the flavor dependence of the interactions, and the SU_{SF} (6) term $(AC_2[SU_{SF}(6)])$ depends on the permutation symmetry of the wave functions, represents "signature-dependent" interactions[33]. The last two terms $(E[C_2[SU_I(2)] - \frac{1}{4}(C_1[U_Y(1)])^2])$ represent the isospin and hypercharge dependence of the masses. The eigenvalues of the Casimir operators in Eq. (15) are:

$$\langle C_2[SU_{SF}(6)]\rangle = \begin{cases} \frac{45}{4} & \text{for}[56] \\ \frac{33}{4} & \text{for}[70] \\ \frac{21}{4} & \text{for}[20] \end{cases}$$

$$\langle C_2[\mathrm{SU}_F(3)]\rangle = \begin{cases} 3 & \text{for}[8] \\ 6 & \text{for}[10] \\ 0 & \text{for}[1] \end{cases}$$

$$\langle C_2[SU_S(2)]\rangle = S(S+1) \qquad \ldots (16)$$

$$\langle C_1[U_Y(1)]\rangle = Y$$

$$\langle C_2[SU_I(2)]\rangle = I(I+1)$$

The generalized Gürsey Radicati mass formula Eq. (15) can be used to describe the octet and decuplet baryons spectrum, provided that two conditions are fulfilled. The first condition is the feasibility of using the same splitting coefficients for different *SU* (6) multiplets. This seems actually to be the case, as shown by the algebraic approach to the baryon spectrum[1]. The second condition is given by the feasibility of getting reliable values for the unperturbed mass values M_0[14]. For this purpose we regarded the *SU* (6) invariant part of the hCQM, which provides a good description of the baryons spectrum and used the Gürsey Radicati inspired *SU* (6) breaking interaction to describe the splitting within each *SU* (6) multiplet. Therefore, the baryons masses are obtained by three quark masses and the eigen energies $(E_{\nu\gamma})$ of the radial Schrödinger equation with the expectation values of H_{GR} as follows:

$$M = 3m + E_{\nu\gamma} + \langle H_{GR}\rangle \qquad \ldots (17)$$

In order to simplify the solving procedure, the constituent quarks masses are assumed to be the same for up, down and strange quark flavors $(m_u = m_d = m_s)$, therefore, within this approximation, the *SU* (6) symmetry is only broken dynamically by the spin and flavour dependent terms in the Hamiltonian. We determined $E_{\nu\gamma}$ by exact solution of the radial Schrödinger equation for the hypercentral Potential Eq. (2). The expectation values of H_{GR}, $(\langle H_{GR}\rangle)$ is identified by Eq. (16). For calculating the baryons mass according to Eq. (15), we need to find the unknown parameters. For this purpose we choose a limited number of well-known resonances and express their mass differences using H_{GR} and the Casimir operator expectation values:

$$\begin{aligned} &N(1650)S11 - N(1535)S11 = 3C \\ &\Sigma(1193)P11 - \Lambda(1116)P01 = 2E \\ &4N(938)P11 - \Sigma(1193)P11 - 3\Lambda(1116)P01 = 4D \end{aligned} \qquad \ldots (18)$$

Leading to the numerical values *C*=38.3, *D*=-197.3 *MeV* and *E*=38.5 *MeV*. We determined *m*, *g* and ω (in Eq. (11)) and the two coefficients *A* and *B* of Eq. (15) in a simultaneous fit to the 3 and 4 star resonances of Table 2 which have been assigned as octet and decuplet states. The fitted parameters are reported in Table 1. The corresponding numerical values are given in table 2, column $M_{ourCalc}$. The percentage of relative error for our calculations is between 0 and 8 % (column 6, in table 2). Comparison between our results and the experimental masses[35] show that the octet and decuplet baryon spectrums are, in general, fairly well reproduced.

Table 1–The fitted values of the parameters of the Eq. (17) for N, Δ, Λ and Σ baryons, obtained with resonances mass differences and global fit to the experimental resonance masses[35]

A	B	*C*	D	*E*	*m*	*g*	*ω*
-17.891 MeV	17.989 MeV	38.300	-197.300 MeV	38.500 MeV	271 MeV	0.289	0.321

Table 2 – Mass spectrum of baryons resonances (in MeV) calculated with the mass formula Eq. (17). The column $M_{Our\ Calc}$ contains our calculations with the parameters of Table 1 and column 6 indicates the percentage of relative error for our calculations

Baryon	Status	Mass(exp)[35]	State	$M_{Our\ Calc}$	Percent of relative error
N(938) P11	****	938	$^{2}8_{1/2}[56, 0^{+}]$	938.00	0%
N(1440) P11	****	1410-1450	$^{2}8_{1/2}[56, 0^{+}]$	1450.31	2.858% - 0.021%
N(1520) D13	****	1510-1520	$^{2}8^{3/2}[70, 1^{-}]$	1524.98	0.992% - 0.320%
N(1535) S11	****	1525-1545	$^{2}8_{1/2}[70, 1^{-}]$	1524.98	0.001% - 1.295%
N(1675) D15	****	1670-1680	$^{4}8_{5/2}[70, 1^{-}]$	1774.43	6.253% - 5.620%
N(1700) D13	***	1650-1750	$^{4}8_{3/2}[70, 1^{-}]$	1774.43	7.541% - 1.396%
N(1710) P11	***	1680-1740	$^{2}8_{1/2}[70, 0^{+}]$	1740.59	3.606% - 0.033%
N(2190) G17	****	2100-2200	$^{2}8_{7/2}[70, 3^{-}]$	2199.08	4.718% - 0.041%
N(2220) H19	****	2200-2300	$^{2}8_{9/2}[56, 4^{+}]$	2280.49	3.658% - 0.848%
N(2250) G19	****	2200-2350	$^{4}8_{9/2}[70, 3^{-}]$	2313.98	5.180% - 1.532%
N(2600) I1,11	***	2550-2750	$^{2}8_{11/2}[70, 5^{-}]$	2739.62	7.436% - 0.377%
Δ (1232) P33	****	1231-1233	$^{4}10_{3/2}[56, 0^{+}]$	1232.37	0.111% - 0.051%
Δ (1620) S31	****	1600-1660	$^{2}10_{1/2}[70, 1^{-}]$	1694.45	5.903% - 2.075%
Δ (1700) D33	****	1670-1750	$^{2}10_{3/2}[70, 1^{-}]$	1694.45	1.464% - 3.174%
Δ (1905) F35	****	1865-1915	$^{4}10_{5/2}[56, 2^{+}]$	1890.23	1.352% - 1.293%
Δ (1910) P31	****	1870-1920	$^{4}10_{1/2}[56, 2^{+}]$	1890.23	1.081% - 1.550%
Δ (1950) F37	****	1915-1950	$^{4}10_{7/2}[56, 2^{+}]$	1890.23	1.293% - 3.065%
Δ (2420) H3, 11	****	2300-2500	$^{4}10_{11/2}[56, 4^{+}]$	2294.76	0.227% - 8.209%
Λ (1116)P01	****	1116	$^{2}8_{1/2}[56, 0^{+}]$	1116.05	0.004%
Λ (1600)P01	***	1560-1700	$^{2}8_{1/2}[56, 0^{+}]$	1649.36	5.728% - 2.978%
Λ (1670)S01	****	1660-1680	$^{2}8_{1/2}[70, 1^{-}]$	1703.03	2.592% - 1.370%
Λ (1690)D03	****	1685-1695	$^{2}8_{3/2}[70, 1^{-}]$	1703.03	1.070%- 0.473%
Λ (1800)S01	***	1720-1850	$^{4}8_{1/2}[70, 1^{-}]$	1817.93	5.693% - 1.733%
Λ (1810)P01	***	1750-1850	$^{2}8_{1/2}[70, 0^{+}]$	1837.58	5.004% - 0.671%
Λ (1820)F05	****	1815-1825	$^{2}8_{5/2}[56, 2^{+}]$	1918.64	5.710% - 5.130%
Λ (1830)D05	****	1810-1830	$^{4}8_{5/2}[70, 1^{-}]$	1817.93	0.438% - 0.659%
Λ (1890)P03	****	1850-1910	$^{2}8_{3/2}[56, 2^{+}]$	1918.64	3.710%- 0.452%
Λ (2110)F05	****	2090-2140	$^{4}8_{5/2}[70, 2^{+}]$	2087.21	0.133% - 2.466%
Λ^{*}(1405) S01	****	1402-1410	$^{2}1_{1/2}[70, 1^{-}]$	1514.78	8.044% - 7.431%
Λ^{*}(1520)D01	****	1518-1520	$^{2}1_{3/2}[70,1^{-}]$	1514.78	0.212% - 0.343%
Σ (1193)P11	****	1193	$^{2}8_{1/2}[56, 0^{+}]$	1193.05	0.004%
Σ (1660)P11	***	1630-1690	$^{2}8_{1/2}[56, 0^{+}]$	1632.33	0.142% - 3.412%
Σ (1670)D13	****	1665-1685	$^{2}8_{3/2}[70, 1^{-}]$	1780.03	6.908% - 5.639%
Σ (1750)S11	***	1730-1800	$^{2}8_{1/2}[70, 1^{-}]$	1780.03	2.891% - 1.109%
Σ (1775)D15	****	1770-1780	$^{4}8_{5/2}[70, 1^{-}]$	1760.65	0.528% - 1.087%
Σ (1915)F15	****	1900-1935	$^{2}8_{5/2}[56, 2^{+}]$	1860.91	2.057% - 3.828%
Σ (1940)D13	***	1900-1950	$^{2}8_{3/2}[56, 1^{-}]$	1928.26	1.487% - 1.114%
Σ^{*}(1385)P13	****	1383-1385	$^{4}10_{3/2}[56, 0^{+}]$	1361.92	1.524% - 1.666%
Σ^{*}(2030)F17	****	2025-2040	$^{4}10_{7/2}[56, 2^{+}]$	2029.78	0.236% - 0.500%

3 Conclusions

In this article a nonrelativistic quark model is used to study the spectrum of the octet and decuplet baryon systems (N, Δ, Λ and Σ baryons). We have solved the Schrödinger equation numerically to obtain the energy eigenvalues under the isotonic oscillator interaction potential. Then, we fitted the generalized GR mass formula parameters to the baryons energies and calculated the baryon masses. The overall good description of the spectrum which we obtain by our proposed model shows that our theoretical model can also be used to give a fair description of the energies of the excited multiplets up to 3 GeV and not only for the ground state octets and decuplet. Moreover, our model reproduces the position of the Roper resonances of the nucleon and negative-parity resonance. There are problems with the reproduction of some hyperons, in particular for the Λ (1670)S01, Λ (1820)F05 and Λ*(1405)S01 the resonances that come out degenerate and above the experimental values. There are still problems in the reproduction of the experimental masses in N (1675)D15, Σ(1915)F15 and Σ (1670)D13 turn out to have predicted mass about 100 MeV above the experimental value. In our calculations the constituent quarks are considered to be the same for up, down and strange quark flavors $(m_u = m_d = m_s)$, within this approximation some errors Occurs. A better agreement may be obtained either using the square of the mass[1] or trying to include a spatial dependence in the *SU* (6)-breaking part.

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