



# Alpha-Nucleon Scattering by Extended Hulthén Potential

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The phase shift analysis of any nucleon-nucleon scattering is an important aspect for both local and non-local potentials to know different properties of any nuclear system. The Phase Function Method (PFM) effectively determines the scattering phase shifts for both local and non local potentials. We use PFM for the local extended Hulthén potential and solve the first order phase equation to generate scattering phase shifts for different states of ( $\alpha$ -n) and ( $\alpha$ -p) systems. We demonstrate the merit of our approach by computing the phase shift data with and without some correction factors and comparing it with standard experimental results.

**Keywords:** Phase function method; Hulthén potential; Extended Hulthén potential; Nucleon-nucleus system; Phase shifts

## 1 Introduction

Hulthén potential<sup>1</sup> is a short ranged potential widely applied in nuclear, particle and atomic physics for its exact solvability for s-wave when used in Schrödinger's equation<sup>2</sup>. Further we can generate approximate solutions for higher partial waves from this s-wave via several mathematical approach- super symmetric factorization methods<sup>3-6</sup> being one of the most used one. This is a local, energy independent exponential type potential which is well fitted for representation of the electromagnetic interaction (Coulomb) for small values of  $r$ , and decreases rapidly at large distances. Hence, the atomic Hulthén potential is termed as the screened Coulomb potential. An extended version of this Hulthén potential has also been proposed by Eğrişes, Demirhan, and Büyükhiliç<sup>7</sup> which has been applied successfully<sup>8,9</sup> for some physical applications. Hall *et al.*<sup>10</sup> in his recent works calculated the exact normalized solutions of the Schrödinger equation for such a deformed extended Hulthén potential.

In this present text we calculate the phase shifts of ( $\alpha$ -n) and ( $\alpha$ -p) scattering for  $\ell = 0, 1$  &  $2$  partial wave states using this Extended Hulthén potential via phase-function method (PFM)<sup>11</sup>. We compare the scattering phase shifts of the aforementioned states with experimental data<sup>12</sup> to study the efficiency of the potential under consideration.

## 2 Extended Hulthén potential

The attractive extended Hulthén potential is expressed as

$$V_q(r) = - \left( \frac{(\beta^2 - \alpha^2) q e^{-(\beta - \alpha)r}}{1 - q e^{-(\beta - \alpha)r}} \right). \quad \dots (1)$$

The terms  $(\beta - \alpha)$  and  $(\beta^2 - \alpha^2)$  stand for the range and depth parameters. The quantities  $\alpha$  and  $\beta$  have the dimensions of inverse length. The quantity  $q$  is a new parameter with condition  $1 > q > 0$ . Similar potential variety has been used with quite success for various applications<sup>7,8</sup>. Extended Hulthén potential is similar to ordinary Hulthén potential as for the shift of  $r \rightarrow r + \log(q/(\beta - \alpha))$  it essentially becomes Hulthén potential of Eckart class<sup>13</sup>. For generation of scattering phase shifts of  $l \neq 0$  partial waves of both ( $\alpha$ -n) and ( $\alpha$ -p) systems, we add an exponentially screened centrifugal barrier term as advocated by several authors<sup>14-17</sup> through supersymmetry (SUSY) formalism with pure Hulthén potential. This is expressed as

$$V_l(r) = \frac{\ell(\ell + 1)}{r^2} \approx (\beta - \alpha)^2 (l(l + 1)) \times \left( \frac{e^{-(\beta - \alpha)r}}{1 - e^{-(\beta - \alpha)r}} \right)^2. \quad \dots (2)$$

The centrifugal barrier term does not explicitly depend on the nature of the potential used. In general, for exponentially screened potentials, a screened centrifugal barrier is always applied for having analytical solutions<sup>18-23</sup>. Thus, a screened barrier is adapted here to deal with the extended Hulthén potential.

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So the effective potential for ( $\alpha$ -n) system yields

$$V_{eff}^{an}(r) = V_q(r) + V_l(r). \quad \dots (3)$$

Putting different  $\ell$  values in  $V_l(r)$  one gets the effective potentials for different states.

For ( $\alpha$ -p) systems, we added atomic Hulthén potential to establish the electromagnetic interaction part

$$V_H = V_0 \frac{e^{-r/a}}{(1 - e^{-r/a})}, \quad \dots (4)$$

where  $V_0$  and  $a$  are two parameters representing strength and screening radius of the potential. For large screening radius, the atomic Hulthén potential behaves as the Coulomb potential such that the product  $V_0 a$  remains constant and one has  $V_0 a = 2k\eta = const$ , where  $\eta$  is the Sommerfeld parameter. Finally, the expression of effective potential for different partial waves of ( $\alpha$ -p) system is expressed as.

$$V_{eff}^{ap}(r) = V_q(r) + V_l(r) + V_H. \quad \dots (5)$$

### 3 Computation of phase shifts

Phase Function Method (PFM) is a technique developed by Calogero<sup>11</sup> to effectively determine the scattering phase shifts for both local and non-local potentials<sup>24-26</sup>. For local potential, like the extended Hulthén plus the atomic Hulthén potential, the phase equation is given by<sup>27-32</sup>

$$\delta'_l(k, r) = -k^{-1} V_{eff}(r) [\cos \delta_l(k, r) \hat{j}_l(kr) - \sin \delta_l(k, r) \hat{\eta}_l(kr)]^2, \quad \dots (6)$$

where  $V_{eff}(r)$  stands for the potential of the respective state. The other functions  $\hat{j}_l(kr)$  and  $\hat{\eta}_l(kr)$  are the Riccati- Bessel functions<sup>33</sup> which take up different values for different  $\ell$ . Solving this first order, non linear differential equation from the origin to asymptotic region with the initial condition  $\delta_l(k, 0) = 0$ , we compute phase shifts for different states of ( $\alpha$ -n) and ( $\alpha$ -p) systems. We then finally compare our phase parameters with standard data and represent our findings in Fig. 1-4.

### 4 Results and discussion

To fix the parameters of different states for both the systems we give free running to our parameters in the numerical routine to reproduce correct phase parameters. The value of  $V_0 a$  for ( $\alpha$ -p) system is  $0.1117 \text{ fm}^{-1}$ . The screening radius  $a$  is considered to be  $20 \text{ fm}$ . The adjustable parameter  $q$  takes up different values for different systems and the values will be mentioned later. With the parameters in Table 1 we have computed elastic scattering phase shifts for various states, represented by  $\delta^{1/2+}$ ,  $\delta^{1/2-}$ ,  $\delta^{3/2-}$ ,  $\delta^{3/2+}$  and  $\delta^{5/2+}$ , using  $\hbar^2 / 2\mu = 25.92 \text{ MeVfm}^2$ , where  $\mu$  is the reduced mass of the system. The phase equation is solved by running  $r$  from zero to  $7 \text{ fm}$  (asymptotic region)

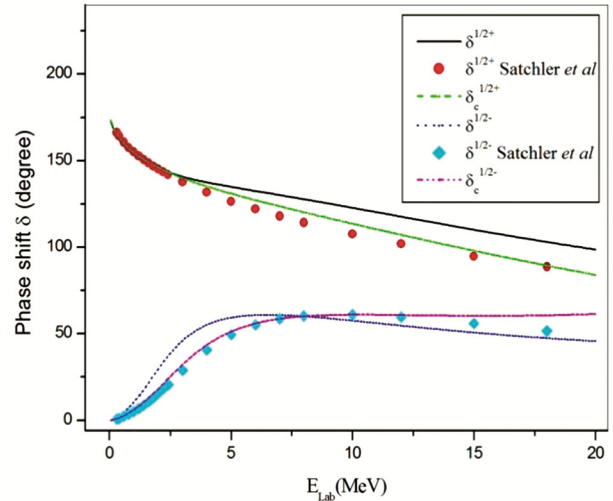


Fig. 1 — Phase shift for ( $\alpha$ -n) 1/2(+) and 1/2(-) states.

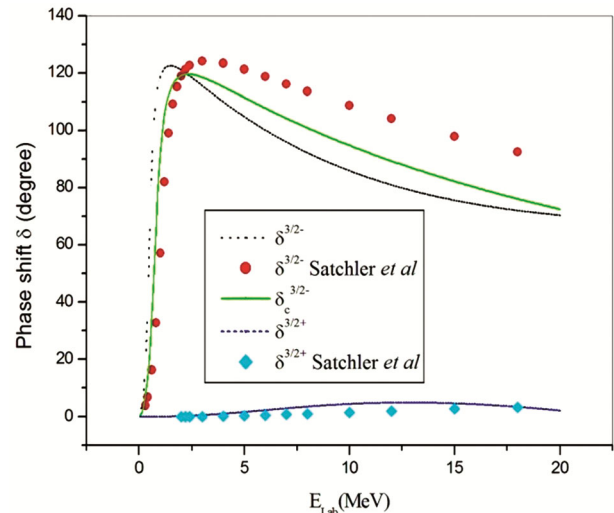


Fig. 2 — Phase shift for ( $\alpha$ -n) 3/2(-) and 3/2(+) state.

in steps of  $0.01 \text{ fm}$ . The saturation in phase shifts is reached beyond  $7 \text{ fm}$ .

In Figs. 1-6 we present our phase data for  $1/2(+)$ ,  $1/2(-)$  and  $3/2(-)$ ,  $3/2(+)$  and  $5/2(+)$  states of  $(\alpha-n)$  and  $(\alpha-p)$  systems for extended Hulthén potential. The newly introduced parameter  $q$  takes the values 0.88,

0.8, 0.895, 0.92, 0.92 and 0.915, 0.856, 0.96, 0.96, 0.8 respectively.

Looking closely into Figs. 1-6 it is noticed that the phase parameters for our proposed extended Hulthén potential model reproduce the qualitative nature of the phase shifts for  $1/2(+)$ ,  $1/2(-)$ ,  $3/2(-)$ ,  $3/2(+)$  and  $5/2(+)$  states of the  $(\alpha-n)$  and  $(\alpha-p)$  systems, respectively. For the  $(\alpha-n)$  system we observe that our phase values  $\delta^{1/2+}$ ,  $\delta^{1/2-}$ ,  $\delta^{3/2-}$  and  $\delta^{5/2+}$  differ more or less symmetrically on either side of  $E_{\text{Lab}} = 2.4 \text{ MeV}$ ,  $E_{\text{Lab}} = 8.72 \text{ MeV}$ ,  $E_{\text{Lab}} = 2.25 \text{ MeV}$  and  $E_{\text{Lab}} = 16.78 \text{ MeV}$ , respectively while those for the  $(\alpha-p)$  system are  $E_{\text{Lab}} = 5.0 \text{ MeV}$ ,  $E_{\text{Lab}} = 6.02 \text{ MeV}$ ,  $E_{\text{Lab}} = 4.7 \text{ MeV}$  and  $E_{\text{Lab}} = 16.53 \text{ MeV}$  respectively.

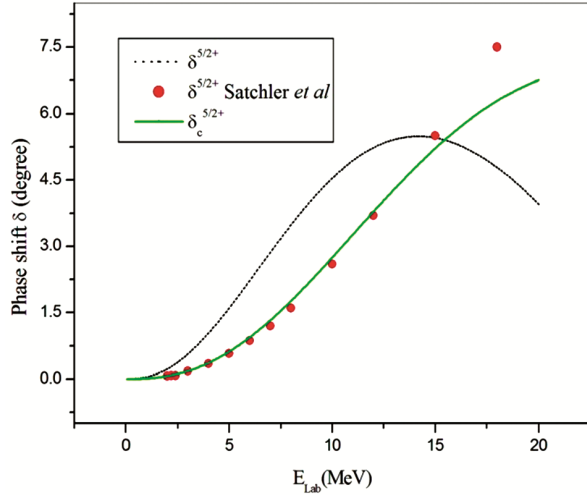


Fig. 3 — Phase shift for  $(\alpha-n)$   $5/2(+)$  state.

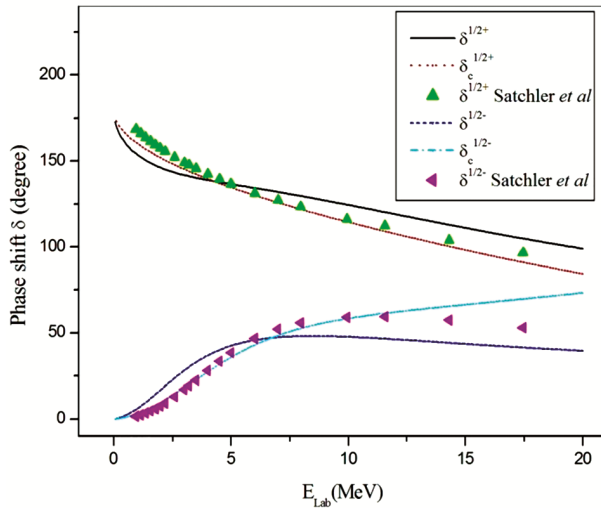


Fig. 4 — Phase shift for  $(\alpha-p)$   $1/2(+)$  and  $1/2(-)$  states

Table 1 — Parameters for the  $(\alpha-n)$  and  $(\alpha-p)$  systems.

States	$(\alpha-n)$ system		$(\alpha-p)$ system	
	$\alpha \text{ fm}^{-1}$	$\beta \text{ fm}^{-1}$	$\alpha \text{ fm}^{-1}$	$\beta \text{ fm}^{-1}$
$1/2(+)$	0.949	1.135	1.011	1.135
$1/2(-)$	1.975	2.52	1.927	2.49
$3/2(-)$	2	2.45	2.47	3.12
$3/2(+)$	1.2	1.26	1.055	1.15
$5/2(+)$	1.25	1.43	1.8	2.05

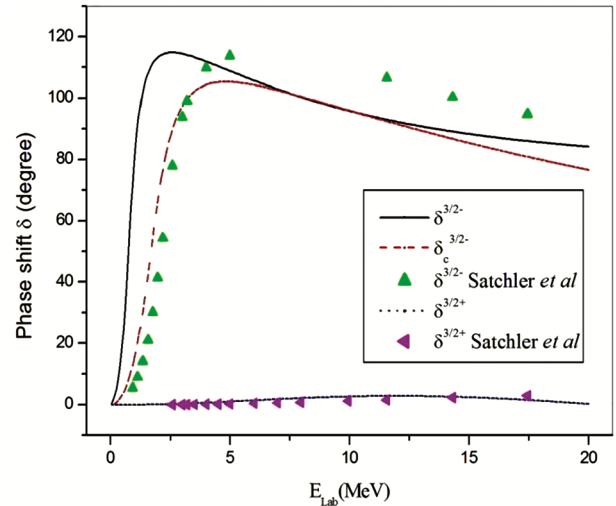


Fig. 5 — Phase shift for  $(\alpha-p)$   $3/2(-)$  and  $3/2(+)$  state.

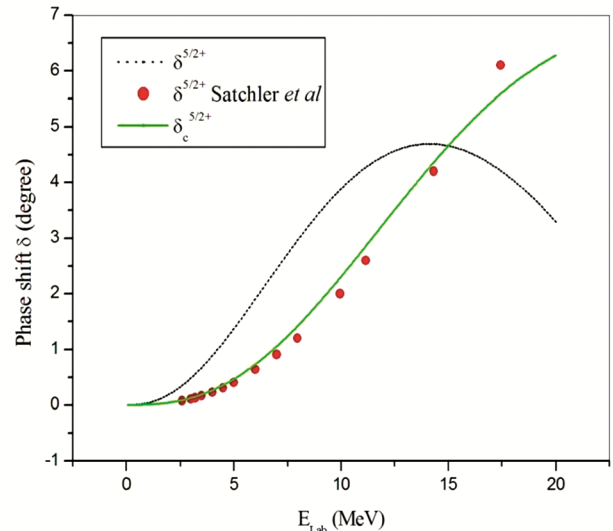


Fig. 6 — Phase shift for  $(\alpha-p)$   $5/2(+)$  state.

However, the phase shifts for  $3/2(+)$  states of both the systems are in good agreement with ref<sup>12</sup>. Therefore, our central potential model needs an energy-dependent correction term on either side of the point of coincidence with experimental data<sup>12</sup> for  $1/2(+)$ ,  $1/2(-)$ ,  $3/2(-)$  and  $5/2(+)$  states to have a better agreement with standard data<sup>12</sup>. To simulate the effect of such correction in the phase data we have identified and incorporated an energy-dependent correction factor to the concerned interactions to achieve good agreement with the experiment<sup>12</sup>. These correction factors read as:

for ( $\alpha$ -n) systems-

$$V_{eff(c)}^{1/2+} = V_{eff}^{1/2+} - 1.25(k^2 - 0.0740) \times \exp(-\gamma(\beta + \alpha)r)$$

$$V_{eff(c)}^{1/2-} = V_{eff}^{1/2-} - 1.25(k^2 - 0.2691) \times \exp(-\gamma(\beta + \alpha)r)$$

$$V_{eff(c)}^{3/2-} = V_{eff}^{3/2-} - 1.25(k^2 - 0.0694) \times \exp(-\gamma(\beta + \alpha)r)$$

$$V_{eff(c)}^{5/2+} = V_{eff}^{5/2+} - 1.25(k^2 - 0.5179) \times \exp(-\gamma(\beta + \alpha)r)$$

for ( $\alpha$ -p) systems-

$$V_{eff(c)}^{1/2+} = V_{eff}^{1/2+} - 1.25(k^2 - 0.2469) \times \exp(-\gamma(\beta + \alpha)r)$$

$$V_{eff(c)}^{1/2-} = V_{eff}^{1/2-} - 1.25(k^2 - 0.1857) \times \exp(-\gamma(\beta + \alpha)r)$$

$$V_{eff(c)}^{3/2-} = V_{eff}^{3/2-} - 1.25(k^2 - 0.145) \times \exp(-\gamma(\beta + \alpha)r)$$

$$V_{eff(c)}^{5/2+} = V_{eff}^{5/2+} - 1.25(k^2 - 0.5102) \times \exp(-\gamma(\beta + \alpha)r).$$

Here  $\gamma$  is an adjustable parameter. For ( $\alpha$ -n) system these are 0.42, 0.12, 0.12, 0.22 and the same for ( $\alpha$ -p) system are 3.6, 1.12, 0.78, 0.15 respectively. The phase parameters with correction factors are also depicted in respective figures and are designated by  $\delta_c^{1/2+}$ ,  $\delta_c^{1/2-}$ ,  $\delta_c^{3/2-}$  and  $\delta_c^{5/2+}$ . Our potential model with correction factors can well account the phase shift values up to 15 MeV for all the s, p and d-states

except the  $3/2-$  state beyond 3.5 MeV. Our peak values for  $3/2-$  state fall below the experimental result by  $5^0$ . Beyond this laboratory energy our phase parameters reproduce slight lower values than the standard data<sup>12</sup>. A variation of about 10% in phase parameters is observed in the case of Ahmad *et al.*<sup>36</sup> while the same is within 8% only in our case from the standard data<sup>12</sup>. The overall agreement of the present results with those of Ahmad *et al.*<sup>36</sup>, Mazur *et al.*<sup>37</sup>, Cattapan *et al.*<sup>38</sup>, Dohet-Eraly *et al.*<sup>39</sup> & Bhoi *et al.*<sup>40</sup> is noteworthy.

We portray the potentials, with and without correction factors, in Figs. 7-11 for both ( $\alpha$ -n) and ( $\alpha$ -p) systems in the energy unit by multiplying the effective potential with the factor  $\hbar^2 / 2\mu$  MeVfm<sup>2</sup>. These are denoted by  $V_{eff}^{1/2+}$ ,  $V_{eff}^{1/2-}$ ,  $V_{eff}^{3/2-}$ ,  $V_{eff}^{3/2+}$ ,  $V_{eff}^{5/2+}$  and  $V_{eff(C)}^{1/2+}$ ,  $V_{eff(C)}^{1/2-}$ ,  $V_{eff(C)}^{3/2-}$ ,  $V_{eff(C)}^{5/2+}$  respectively. For clarity of presentation the potentials are plotted up to 5.5 fm. It is observed that repulsive cores develop in our potentials for various partial wave states except  $1/2(+)$  state of the ( $\alpha$ -n) system. The s-wave potential for the ( $\alpha$ -n) system is purely attractive in nature while the same for the ( $\alpha$ -p) system possesses a hard core followed by a strong attractive part. This quasi hard core originates due to repulsive electromagnetic interaction. The p and d-wave potentials for both ( $\alpha$ -n) and ( $\alpha$ -p) systems possess repulsive cores due to addition of centrifugal barriers as well as electromagnetic interaction followed by attractive parts. With the addition of energy-dependent correction factors to the respective potentials the depth of the potentials alter to fit proper phase data to

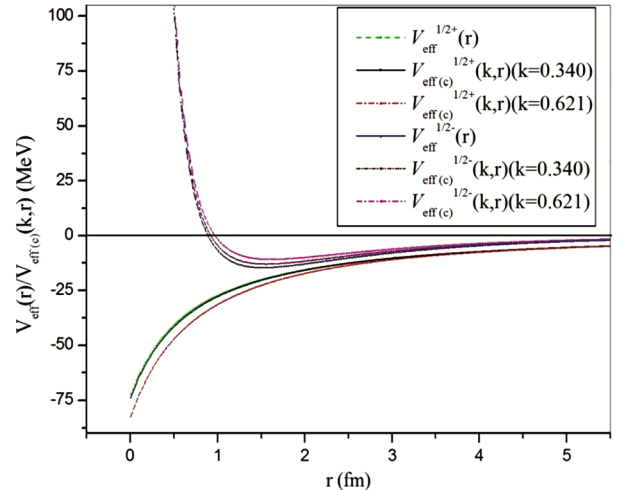
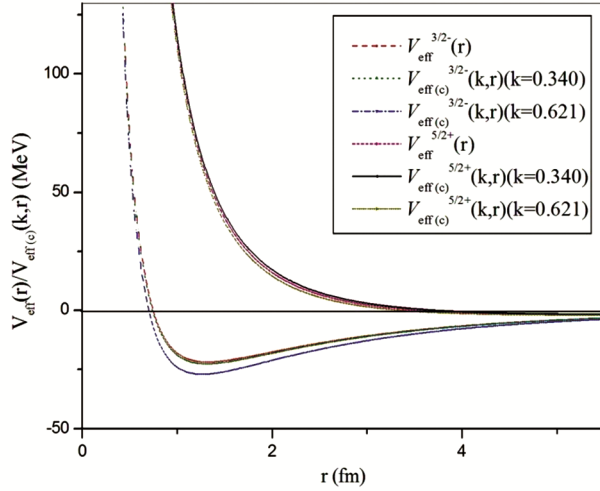
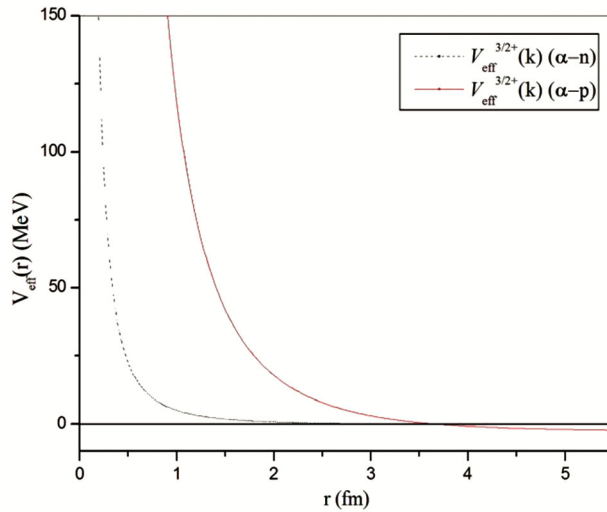
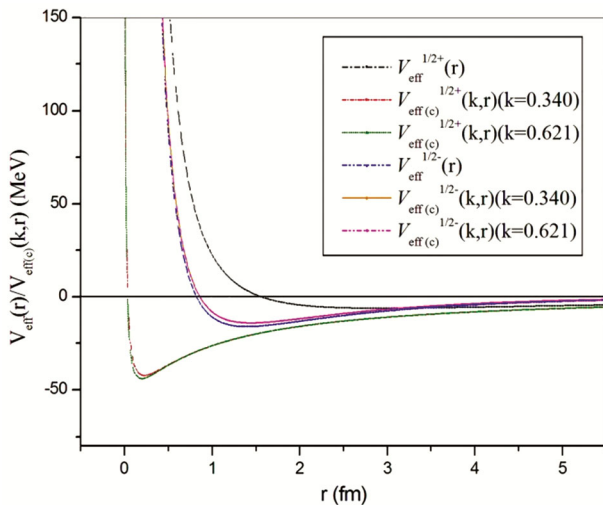
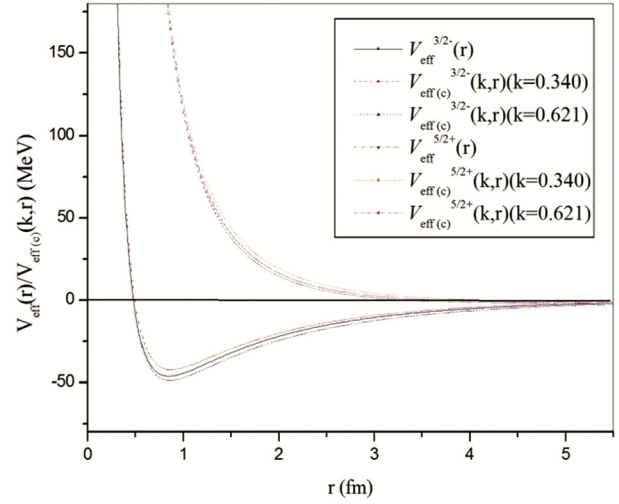


Fig. 7 — Potentials for ( $\alpha$ -n) system  $1/2(+)$  and  $1/2(-)$  states.


 Fig. 8 — Potentials for ( $\alpha$ -n) system  $3/2(-)$  and  $5/2(+)$  states.

 Fig. 9 — Potentials for ( $\alpha$ -n) and ( $\alpha$ -p) systems  $3/2(+)$  state.

 Fig. 10 — Potentials for ( $\alpha$ -p) system  $1/2(+)$  and  $1/2(-)$  states.

 Fig. 11 — Potentials for ( $\alpha$ -p) system  $3/2(-)$  and  $5/2(+)$  states.

produce a fairly good agreement with experimental phase shift values<sup>12</sup>. Our observation is fully consistent with the findings of Reichstein & Tang<sup>41</sup> in studying the effective local potential of ( $\alpha$ -n) system obtained from a nonlocal one in microscopic studies. However, in this study, the hard cores do not possess too much physical significance as we are concerned with the low energy scattering where the tail parts of these potentials play an important role.

## 5 Conclusions

In the present text we have calculated of scattering phase shifts of different states ( $1/2(+)$ ,  $1/2(-)$ ,  $3/2(+)$ ,  $3/2(-)$  and  $5/2(+)$  for both ( $\alpha$ -n) and ( $\alpha$ -p) systems using extended Hulthén potential as the nuclear part of the interaction without any spin-orbit and tensor interactions. The s, p and d wave potentials for both the systems with suitable  $q$  values show correct trends of nuclear potentials. Quasi hard cores develop in the potentials except for the  $1/2(+)$  state of the ( $\alpha$ -n) system and die out gradually with the distance. With an additional energy-dependent correction factor to the potential good quantitative agreement in the phase parameters was achieved. Thus, the energy-dependent correction factors to the interactions, to some extent, have the ability to reproduce the effects of the non-central parts of the nuclear interactions. The  $\text{He}^5$  system (likewise  $\text{Li}^5$ ) is an unbound system, its ground state being a narrow  $p$ -wave resonance in the  $3/2^-$  and  $1/2^-$  channel. A critical phase shift analysis of Hoop & Barshall<sup>42</sup> regarding ( $\alpha$ -n) scattering show two resonances namely  $1/2^-$  and  $3/2^-$  at 1 and 4 MeV. These two states are, in fact, the meta-stable states of

the ( $\alpha$ -n) system. Our Hamiltonian contains only nuclear and electromagnetic central potentials with an energy-dependent term. On the other hand, Quaglioni & Navrátil<sup>43</sup> calculated (n- $\alpha$ ) and (p- $\alpha$ ) phase shifts by combining the resonating-group method and a microscopic description of the nucleon clusters with more realistic N<sup>3</sup>LO and CD-Bonn  $NN$  potentials while Lee & Robson<sup>44</sup>, in the folding model approach, also generated a nucleon-nucleus optical potential with the inclusion of spin-orbit and tensor forces. They achieved an excellent description of alpha-nucleon s-wave phase shifts while those for the  $p$ -wave the same have insufficient magnitude and splitting with respect to experimental results<sup>12</sup>. Our  $p$ -wave results reproduce much better results than those of Ref.<sup>37</sup> and are in reasonable agreement with the experimental data<sup>12</sup>. The overall quality of the consistency between the theoretical and experimental data, in the energy region under consideration, is noteworthy.

We conclude by noting that our three parameter central potential with an energy-dependent correction factor in conjunction with phase function method may be good enough to treat nucleon-nucleus or light nucleus-nucleus systems.

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