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Pulsatile Flow Through a Curved Pipe – An Analytical Study

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In this paper, the unsteady flow of Newtonian fluid through a curved pipe due to a pulsatile pressure gradient has been considered. The flow is three dimensional and the partial differential equations governing the flow are highly coupled and non-linear. Approximate analytical solutions of the governing partial differential equations have been obtained without neglecting any term containing the curvature ratio. Perturbation series in terms of curvature ratio has been used for obtaining the solutions. It is interesting to note that the solutions are valid for large as well as small values of Womersley number. The effect of different parameters such as, Reynolds number, Womersley number and curvature ratio on the flow through curved pipe is discussed in this paper. It is found that the axial velocity is qualitatively periodic in nature, as expected. The Reynolds number and curvature ratio are found to shift the axial velocity towards the outer boundary of the curved pipe.

Keywords: Pulsatile flow; Curvature ratio; Centrifugal force; Axial velocity; Circular cross-section

1 Introduction

The study of the flow through curved pipes is interesting and challenging because of its flow geometry and the complicated nature of the underlying governing partial differential equations. The maximum velocity shifts towards outer boundary of pipe due to arising secondary flow. In this case, centrifugal force sets-up due to the curvature of pipe and it is balanced by a pressure gradient directing towards the centre of curvature. Thomson[1], for the first time, gave the experimental explanation of the flow in curved systems. Grindley & Gibson[2], Eustice[3,4], White[5] and Taylor[6] had also undertaken the experimental investigation on the streamline motion in curved pipe. Dean[7,8] was the first to work on the study of Newtonian fluid flow through curved pipes, mathematically, using a parameter $K=\frac{2W_0^2a^3}{\upsilon^3R}$, which was later termed as Dean Number. Barua[9], McConalogue & Srivastava[10] and Greenspan[11] have extended Dean's work for large Dean number, although the expressions they had used for Dean number in their study were not unique.

Lyne[12] studied the unsteady flow through curved pipes for small values of Womersley number. However, this work did not account all the effects of curvature as it has neglected few terms involving curvature ratio. Zalosh & Nelson[13] analyzed fully developed laminar flow in a curved tube of circular cross-section under the influence of a pressure gradient that is oscillating sinusoidally in time. Later, Hamakiotes & Berger[14], Sudo *et al.*[15] studied pulsalite flow in curved pipes for large range of Dean number, using different numerical methods. Chang & Tarbell[16], Swanson *et al.*[17] did the experimental investigation on the pulsatile flow in a curved pipe.

Before 2008, a majority of studies on the pulsatile flow through curved pipes were considered only for weakly curved pipes of circular cross section, by neglecting curvature ratio terms. Siggers & Waters[18] worked on the unsteady fluid flow in a curved pipe without ignoring terms containing curvature ratio. They have obtained analytical solutions only for smaller values of Dean and Womersley numbers, and the solutions for higher values of Dean and Womersley numbers were obtained numerically. In this paper, we have studied the flow of Newtonian fluid through a curved pipe due to a pulsatile pressure gradient. Governing partial differential equations are considered without neglecting any term containing the curvature ratio for the sake of simplicity, and the analytical solutions are obtained. The solutions are valid for any value of curvature ratio and Womersley number.

2 Mathematical Formulation

Consider an unsteady Newtonian fluid flow through a curve pipe due to a pulsatile pressure gradient,

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$\frac{-\partial p'}{\partial s'} = G + \sum_{n=1}^{\infty}\left(p_{cn}\cos(\omega n t') + p_{sn}\sin(\omega n t')\right)$, where, G is the steady part of the pressure gradient, p_{cn} and p_{sn} represent the cosine and sine amplitudes of a harmonic forcing function, respectively. Curvature of the pipe is k, where, $k=1/R$ and R is the distance between the center line of the pipe to the center of the axis, say Y-axis, and a is the radius of circular cross section of the pipe, s' is the center-line. We use the coordinates, (r', θ, s') to locate any point in the pipe at time t'. The flow geometry is as shown in Fig. 1. Assuming that the flow is symmetric, it is independent of s' and taking the fluid velocity $\bar{u} = (u', v', w')$, the equations governing the flow are given by Pedley[19],

Continuity equation:

$$\frac{\partial u'}{\partial r'} + \frac{u'}{r'} + \frac{1}{r'}\frac{\partial v'}{\partial \theta} + \frac{k}{h}(u'\cos\theta - v'\sin\theta) = 0, \qquad \ldots (1)$$

Momentum equations:

$$\rho\left[\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial r'} + \frac{v'}{r'}\frac{\partial u'}{\partial \theta} - \frac{v'^2}{r'} - \frac{w'^2 k\cos\theta}{h'}\right] = -\frac{\partial p'}{\partial r'} - \frac{\mu}{r'h}\frac{\partial}{\partial \theta}\left[h\left(\frac{\partial v'}{\partial r'} + \frac{v'}{r'} - \frac{1}{r'}\frac{\partial u'}{\partial \theta}\right)\right], \qquad \ldots (2)$$

$$\rho\left[\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial r'} + \frac{v'}{r'}\frac{\partial v'}{\partial \theta} + \frac{u'v'}{r'} + \frac{w'^2 k\sin\theta}{h}\right] = \frac{-1}{r'}\frac{\partial p'}{\partial \theta} - \frac{\mu}{h}\frac{\partial}{\partial r'}\left[h\left(\frac{\partial v'}{\partial r'} + \frac{v'}{r'} - \frac{1}{r'}\frac{\partial u'}{\partial \theta}\right)\right], \qquad \ldots (3)$$

$$\rho\left[\frac{\partial w'}{\partial t'} + u'\frac{\partial w'}{\partial r'} + \frac{v'}{r'}\frac{\partial w'}{\partial \theta} + \frac{kw'}{h}(u'\cos\theta - v'\sin\theta)\right] = \frac{-1}{h}\frac{\partial p'}{\partial s'} + \frac{\mu}{r'}\left[\frac{\partial}{\partial r'}\left(\frac{r'}{h}\frac{\partial}{\partial r'}(hw')\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r'h}\frac{\partial}{\partial \theta}(hw')\right)\right], \qquad \ldots (4)$$

where, $h = 1 + kr\cos\theta$.

As the characteristic velocity is not a priori and the flow is due to the pulsatile pressure gradient, the steady part of the pressure gradient (G) is used for non-dimensionalising the equations governing the flow. Using the non-dimensional variables,

$$r = \frac{r'}{a}, s = \frac{s'}{a}, u = \frac{a\rho}{\mu}u', v = \frac{a\rho}{\mu}v', w = \frac{\mu}{Ga^2}w', p = \frac{1}{aG}p', t = t'\omega$$

in equations (1)-(4), we obtain non-dimensional equations as,

Continuity equation:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\delta}{H}(u\cos\theta - v\sin\theta) = 0, \qquad \ldots (5)$$

Momentum equations:

$$\alpha^2\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} - \frac{\delta \mathrm{Re}^2 w^2\cos\theta}{H} = -\mathrm{Re}\frac{\partial p}{\partial r} - \frac{1}{H}\frac{\partial}{\partial \theta}\left[H\left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\right], \qquad \ldots (6)$$

$$\alpha^2\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} + \frac{\delta \mathrm{Re}^2 w^2\sin\theta}{H} = -\frac{\mathrm{Re}}{r}\frac{\partial p}{\partial \theta} + \frac{1}{H}\left[H\left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\right], \qquad \ldots (7)$$

$$\alpha^2\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{\delta w}{H}\begin{pmatrix} u\cos\theta \\ -v\sin\theta \end{pmatrix} = -\frac{1}{H}\frac{\partial p}{\partial s} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{r}{h}\frac{\partial}{\partial r}(Hw)\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{rh}\frac{\partial}{\partial \theta}(Hw)\right)\right], \qquad \ldots (8)$$

in which, $\alpha^2 = \frac{a^2\omega}{\upsilon}$ is Womersley number, $\mathrm{Re} = \frac{a^3\rho^3 G}{\mu^2}$ is the Reynolds number and $\delta = \frac{a}{R}$ is the curvature ratio and $H = 1 + \delta r\cos\theta$.

It is to note here that, from the geometry of the problem, the curvature ratio δ approaches to zero corresponds to the flow through straight pipe. It follows immediately that, theoretically, supposing $\delta(= a/R)$ very small in the motion of curved pipe approximates the motion of straight pipe.

We have to solve the non-dimensional equations (5)-(8) together with the following non-dimensional boundary conditions:

No-slip conditions: (u,v,w)= (0,0,0) on $r = 1$,

Natural conditions: u,v,w are finite at $r = 0$,

Symmetry conditions:

$$\frac{\partial u}{\partial \theta} = 0 = \frac{\partial w}{\partial \theta} = v(r,\theta,t) \text{ at } \theta = 0, \pi.$$

Now, eliminating pressure gradient terms from

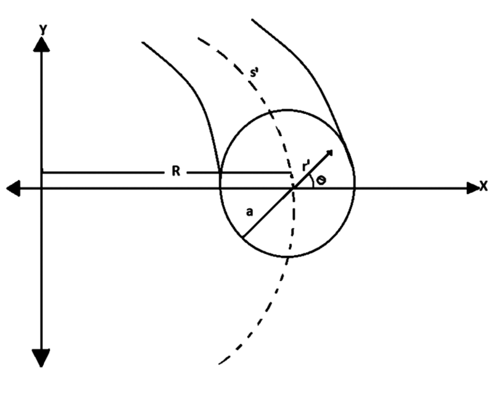


Fig. 1 — Geometry of the flow.

equations (6) and (7), we get,

$$\alpha^2\frac{\partial^2 u}{\partial\theta\partial t}-\alpha^2\frac{\partial v}{\partial t}-\alpha^2 r\frac{\partial^2 v}{\partial r\partial t}+\frac{\partial u}{\partial r}\frac{\partial u}{\partial\theta}$$
$$+u\frac{\partial^2 u}{\partial r\partial\theta}+\frac{v}{r}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial v}{\partial\theta}\frac{\partial u}{\partial\theta}$$
$$-\frac{2v}{r}\frac{\partial v}{\partial\theta}-2u\frac{\partial v}{\partial r}-r\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}-ru\frac{\partial^2 v}{\partial r^2} \quad \ldots (9)$$
$$-\frac{\partial v}{\partial r}\frac{\partial v}{\partial\theta}-v\frac{\partial^2 v}{\partial r\partial\theta}-v\frac{\partial u}{\partial r}$$
$$-2\delta\,\mathrm{Re}^2\frac{w}{H}\left(\cos\theta\frac{\partial w}{\partial\theta}+r\sin\theta\frac{\partial w}{\partial r}\right)$$
$$=\frac{-1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{H}\frac{\partial}{\partial\theta}\left(H\left(\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta}\right)\right)\right)$$
$$-\frac{\partial}{\partial r}\left(\frac{r}{H}\frac{\partial}{\partial r}\left(H\left(\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial\theta}\right)\right)\right)$$

To avoid the complexity in solving equations (5), (8) and (9), in view of the axi-symmetric nature of the flow, we introduce the stream function ψ through $u=\frac{1}{Hr}\frac{\partial(H\psi)}{\partial\theta}, v=\frac{-1}{H}\frac{\partial(H\psi)}{\partial r}$. This stream function ψ satisfies the continuity equation (5), and equations (8) and (9) become,

$$\alpha^2\frac{\partial\psi}{\partial t}+\frac{1}{Hr}\mathrm{F}\left(\frac{\nabla^2\psi}{H},H\psi\right)-2\delta\,\mathrm{Re}^2\frac{w}{H}\mathrm{T}(w)=\nabla^4\psi \quad \ldots (10)$$

$$\alpha^2\frac{\partial w}{\partial t}+\frac{1}{Hr}\mathrm{F}(w,H\psi)+\frac{\delta w}{H}\mathrm{T}(\psi)=\frac{1}{H}+\nabla^2 w \quad \ldots (11)$$

where,

$$\nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{H}\frac{\partial(Hf)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{H}\frac{\partial(Hf)}{\partial\theta}\right),$$
$$\mathrm{F}(f,g)=\frac{\partial f}{\partial r}\frac{\partial g}{\partial\theta}-\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial r},$$
$$\mathrm{T}(f)=\sin\theta\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}$$

for any scalar functions *f* and *g*.

It is to be mentioned here that, equations (10)-(11) have been derived without any assumption on the curvature of the pipe and without neglecting any term containing the curvature ratio for the sake of simplification. As a result, there present an additional (third) term on the left-hand side of equation (11); this was neglected by many researchers. The described term of equation (11), which is due to the centrifugal force arising from the curvature, superimpose the effect of secondary motion into the primary flow. Therefore, the authors are of the opinion that, the aforementioned equations (10)-(11) governing the flow are comparatively more general.

The boundary conditions turn out to be,

No-slip conditions:

$$(1+\delta\cos\theta)\frac{\partial\psi}{\partial\theta}=\delta\psi\sin\theta,\ (1+\delta\cos\theta)\frac{\partial\psi}{\partial r}=-\delta\psi\cos\theta,$$
$$w(r,\theta)=0 \qquad \text{on } r=1,\ t>0,$$

Natural conditions:

$$\psi,w(r,\theta)\rightarrow \text{finite as } r\rightarrow 0,\ t>0,$$

Symmetry conditions:

$$(1+\delta r)\frac{\partial^2\psi}{\partial\theta^2}=\delta r,\ (1+\delta r)\frac{\partial^2\psi}{\partial r^2}=-\delta\psi \text{ at } \theta=0, t>0,$$
$$(1-\delta r)\frac{\partial^2\psi}{\partial\theta^2}=-\delta r,\ (1-\delta r)\frac{\partial^2\psi}{\partial r^2}=\delta\psi \text{ at } \theta=\pi, t>0,$$
$$\frac{\partial w}{\partial\theta}=0 \text{ at } \theta=0,\pi,t>0. \quad \ldots (12)$$

The only non-dimensional parameters present in the boundary value problem (10)-(12) concerning the flow through curved pipe under consideration are, α, Re and δ. As it has been mentioned by Berger *et. al.*[20], the parameter δ is more detailed measure of the effect of geometry and affects the balance of inertia, viscous, and centrifugal forces; it plays a major role in curved-pipe flows.

It is challenging to solve the system (10)-(11), with the conditions given in equation (12). As the problem is not amenable for the exact solutions, an attempt is made to find approximate analytical solutions using perturbation series.

3 Method of solution

We seek the solution of equations (10)-(12) in the form of perturbation series,

$$\psi=\sum_{K=0}^{\infty}\delta^K\psi_K, w=\sum_{K=0}^{\infty}\delta^K wK \quad \ldots (13)$$

After substituting (13) in equations (10)-(11) and comparing the like powers of δ, the zeroth order and first order systems are obtained as,

Zeroth order system:

$$\alpha^2\frac{\partial^2\psi_0}{\partial t\partial\theta}-\alpha^2\frac{\partial\psi_0}{\partial t}-\alpha^2 r\frac{\partial^2\psi_0}{\partial r\partial t}-\frac{1}{r^3}\frac{\partial\psi_0}{\partial\theta}\frac{\partial^2\psi_0}{\partial\theta^2}$$
$$+\frac{1}{r^2}\frac{\partial\psi_0}{\partial\theta}\frac{\partial^3\psi_0}{\partial r\partial\theta^2}-\frac{1}{r^2}\frac{\partial\psi_0}{\partial r}\frac{\partial^3\psi_0}{\partial\theta^3}-\frac{1}{r^3}\frac{\partial^3\psi_0}{\partial\theta^3}$$
$$-\frac{2}{r}\frac{\partial\psi_0}{\partial r}\frac{\partial^2\psi_0}{\partial\theta^2}+\frac{2}{r}\frac{\partial\psi_0}{\partial\theta}\frac{\partial^2\psi_0}{\partial r^2}+\frac{\partial\psi_0}{\partial\theta}\frac{\partial^2\psi_0}{\partial r^2}$$
$$-\frac{1}{r}\frac{\partial^2\psi_0}{\partial r^2}\frac{\partial^2\psi_0}{\partial r\partial\theta}+\frac{\partial\psi_0}{\partial\theta}\frac{\partial^3\psi_0}{\partial r^3}-\frac{\partial^2\psi_0}{\partial r^2}\frac{\partial^2\psi_0}{\partial r\partial\theta}$$
$$-\frac{\partial\psi_0}{\partial r}\frac{\partial^3\psi_0}{\partial r^2\partial\theta}+\frac{1}{r}\frac{\partial\psi_0}{\partial r}\frac{\partial^2\psi_0}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\psi_0}{\partial r}\frac{\partial^3\psi_0}{\partial r\partial\theta}$$
$$=r\left(\frac{\partial^2\psi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi_0}{\partial\theta^2}\right), \quad \ldots (14)$$

$$\alpha^2\frac{\partial w_0}{\partial t}+\frac{1}{r}\left[\frac{\partial w_0}{\partial r}\frac{\partial \psi_0}{\partial \theta}-\frac{\partial w_0}{\partial \theta}\frac{\partial \psi_0}{\partial r}\right]$$
$$=\frac{-\partial p}{\partial s}+\frac{\partial^2 w_0}{\partial r^2}+\frac{1}{r}\frac{\partial w_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2}. \quad \ldots (15)$$

First order system:

$$\alpha^2\frac{\partial}{\partial t}\left[\frac{\partial^2\psi_1}{\partial r^2}+\frac{1}{r}\frac{\partial \psi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi_1}{\partial \theta^2}\right]-2Re^2 w_0 r\sin\theta\frac{\partial w_0}{\partial r}$$
$$=\left[\frac{1}{r}\frac{\partial^4\psi_1}{\partial r^2\partial\theta^2}+\frac{1}{r^2}\frac{\partial^3\psi_1}{\partial r\partial\theta^2}+\frac{1}{r^3}\frac{\partial^4\psi_1}{\partial\theta^4}+2\frac{\partial^3\psi_1}{\partial r^3}+\frac{1}{r^2}\frac{\partial\psi_1}{\partial r}\right]$$
$$-\left[\frac{1}{r}\frac{\partial^2\psi_1}{\partial r^2}-\frac{3}{r^2}\frac{\partial^3\psi_1}{\partial r\partial\theta^2}+\frac{1}{r}\frac{\partial^4\psi_1}{\partial r^2\partial\theta^2}+\frac{4}{r^3}\frac{\partial^2\psi_1}{\partial\theta^2}+r\frac{\partial^4\psi_1}{\partial r^4}\right] \quad \ldots(16)$$

$$\alpha^2\frac{\partial w_1}{\partial t}+\frac{1}{r}\frac{\partial \psi_1}{\partial \theta}\frac{\partial w_0}{\partial r}-\frac{1}{r}\frac{\partial \psi_1}{\partial r}\frac{\partial w_0}{\partial \theta}=-r\cos\theta\frac{\partial p}{\partial s}$$
$$+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_1}{\partial r}+rw_0\cos\theta\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial w_1}{\partial\theta}-rw_0\sin\theta\right). \quad \ldots(17)$$

As the limiting case $\delta\to 0$ corresponds to the flow through straight pipe, we take the solution of equation (14) to be $\psi_0=0$ and equation (15) becomes,

$$\alpha^2\frac{\partial w_0}{\partial t}=\frac{-\partial p}{\partial s}+\frac{\partial^2 w_0}{\partial r^2}+\frac{1}{r}\frac{\partial w_0}{\partial r}. \quad \ldots (18)$$

Since the flow under consideration is pulsatile in nature, following the technique adapted by Majdalani & Chibli[21], we take, $\frac{-\partial p}{\partial s}=1+\sum_{n=1}^{\infty}p_n e^{int}$, and seek the solution of equation (18) as,

$$w_0=w_{00}+\sum_{n=1}^{\infty}w_{0n}e^{int}, \quad \ldots (19)$$

where, $p_n=p_{cn}-ip_{sn}$ and $w_{0n}=w_{0cn}-iw_{0sn}$.

Note that the real part of equation (19) gives the solution of equation (18).

Making use of equation (19) in equation (18), we get,

$$\sum_{n=1}^{\infty}ine^{int}w_{on}=(1+\sum_{n=1}^{\infty}p_n e^{int})+\frac{1}{r}\frac{dw_{00}}{dr}$$
$$+\frac{d^2w_{00}}{dr^2}+\frac{1}{r}\sum_{n=1}^{\infty}\frac{dw_{0n}}{dr}e^{int}+\sum_{n=1}^{\infty}\frac{d^2w_{0n}}{dr^2}e^{int}. \quad \ldots (20)$$

Comparing the coefficients of e^0 from equation (20), we get,

$\frac{d^2w_{00}}{dr^2}+\frac{1}{r}\frac{dw_{00}}{dr}+1=0$, whose solution is given by,

$$w_{00}=\frac{(1-r^2)}{4}. \quad \ldots(21)$$

Comparing the coefficients of e^{int} from Eq. (20), we get,

$\frac{d^2w_{0n}}{dr^2}+\frac{1}{r}\frac{dw_{0n}}{dr}+p_n=\alpha^2 nw_{0n}$, whose solution is obtained as,

$$w_{0n}=\frac{p_n}{\alpha^2}+C_1J_0(i^{3/2}n^{1/2}\alpha r)+C_2Y_0(i^{3/2}n^{1/2}\alpha r),$$

where, J_0 and Y_0 are zeroth order Bessel functions of first and second kind, respectively, C_1 and C_2 are arbitrary constants.

Due to the natural condition, as Y_0 approaches to infinity when r approaches to 0, we take $C_2=0$, to get,

$$w_{0n}=\frac{p_n}{in\alpha^2}+C_1J_0(i^{3/2}n^{1/2}\alpha r).$$

Using the no-slip condition, we get, $C_1=\frac{-p_n}{in\alpha^2}\frac{1}{J_0(\alpha i^{3/2}n^{1/2})}$ and hence

$$w_{0n}=\frac{p_n}{in\alpha^2}\left[1-\frac{J_0(i^{3/2}n^{1/2}\alpha r)}{J_0(i^{3/2}n^{1/2}\alpha)}\right] \quad \ldots (22)$$

Using Eq. (21) and Eq. (22) in Eq. (19), we get,

$$w_0=\frac{1}{4}(1-r^2)+\sum_{n=1}^{\infty}\frac{p_n}{in\alpha^2}\left[1-\frac{J_0(i^{3/2}n^{1/2}\alpha r)}{J_0(i^{3/2}n^{1/2}\alpha)}\right]e^{int} \quad \ldots (23)$$

Now, we find the solution of equations (16)-(17) by seeking,

$$\psi_1=\psi_{10}+\sum_{n=1}^{\infty}\left(\psi_{1cn}\cos(nt)+\psi_{1sn}\sin(nt)\right), \text{ and}$$

$$w_1=w_{10}+\sum_{n=1}^{\infty}\left(w_{1cn}\cos(nt)+w_{1sn}\sin(nt)\right).$$

Proceeding as in the case of zeroth order, taking

$\psi_1 = \psi_{10} + \sum_{n=1}^{\infty} \psi_{1n} e^{int}$ and $w_1 = w_{10} + \sum_{n=1}^{\infty} w_{1n} e^{int}$,

where, $\psi_{1n} = \psi_{1cn} - i\psi_{1sn}$ and $w_{1n} = w_{1cn} - iw_{1sn}$ in equations (16) and (17), respectively, the comparison of coefficients of e^0 and e^{it} gives,

$$\nabla^4 \psi_{10} = \frac{\mathrm{Re}^2 r(1-r^2)\sin\theta}{4}, \quad \ldots (24)$$

$$\begin{aligned} \nabla^4 \psi_{11} - i\alpha^2 \nabla^2 \psi_{11} &= \frac{\mathrm{Re}^2 p_1 \sqrt{i}}{2\alpha} \frac{J_1(i^{3/2}\alpha r)}{J_0(i^{3/2}\alpha)}(r^2 - 1) \\ &+ \frac{\mathrm{Re}^2 p_1 r \sin\theta}{i\alpha^2}\left(1 - \frac{J_0(i^{3/2}\alpha r)}{J_0(i^{3/2}\alpha)}\right), \end{aligned} \quad \ldots (25)$$

$$\begin{aligned} &\frac{1}{r}\frac{\partial \psi_{10}}{\partial \theta}\frac{\partial w_{00}}{\partial r} - \frac{1}{r}\frac{\partial \psi_{10}}{\partial r}\frac{\partial w_{00}}{\partial \theta} = -r\cos\theta \\ &+ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_{10}}{\partial r} + rw_{00}\cos\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial w_{10}}{\partial \theta} - rw_{00}\sin\theta\right), \end{aligned} \quad \ldots (26)$$

$$\begin{aligned} &\alpha^2 w_{11} + \frac{1}{r}\frac{\partial \psi_{11}}{\partial \theta}\frac{\partial w_{10}}{\partial r} - \frac{1}{r}\frac{\partial \psi_{11}}{\partial r}\frac{\partial w_{10}}{\partial \theta} = -r\cos\theta \\ &+ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_{11}}{\partial r} + rw_{10}\cos\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial w_{11}}{\partial \theta} - rw_{10}\sin\theta\right). \end{aligned} \quad \ldots (27)$$

The solutions of equations (24)-(27) are obtained as,

$$\psi_{10} = \frac{\mathrm{Re}^2 \sin\theta}{4608}\left(4r - 9r^3 + 6r^5 - r^7\right), \quad \ldots (28)$$

$$\begin{aligned} \psi_{11} = &\frac{\mathrm{Re}^2 p_1 \sin\theta}{i\alpha^2 J_0(i^{3/2}\alpha)}\Bigg[\left(\frac{r^5}{160} + \frac{r^3}{15i\alpha^2} - \frac{r^3}{96}\right)J_0(i^{3/2}\alpha r) \\ &+ \left(\frac{i^{1/2}r^4}{20\alpha} + \left(\frac{2}{15i^{1/2}\alpha^3} - \frac{i^{1/2}}{48\alpha}\right)r^2\right)J_1(i^{3/2}\alpha r)\Bigg] \\ &+ \frac{\mathrm{Re}^2 p_1}{\alpha^4}\left(\frac{r^3}{8} + \frac{r}{i\alpha^2}\right)\sin\theta + \frac{r}{i\alpha^2} S\sin\theta \\ &+ \frac{2J_1(-i^{3/2}\alpha r)\sin\theta}{i^{3/2}\alpha\left(J_2(i^{3/2}\alpha) - J_0(i^{3/2}\alpha)\right) + 2J_1(i^{3/2}\alpha)}\Bigg[-\frac{\mathrm{Re}^2 p_1}{4\alpha^4} \\ &+ \frac{\mathrm{Re}^2 p_1}{i\alpha^2}\left(\frac{1}{96} + \frac{i}{15\alpha^2}\right) + \frac{\mathrm{Re}^2 p_1}{i^{1/2}\alpha}\frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\Bigg(\frac{-2i}{5\alpha^4} \\ &- \frac{i}{240} - \frac{1}{16\alpha^2}\Bigg) + \frac{\mathrm{Re}^2 p_1}{i\alpha^2}\frac{J_2(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\left(\frac{-7}{480} - \frac{1}{15i\alpha^2}\right)\Bigg] \end{aligned} \quad \ldots (29)$$

$$\begin{aligned} w_{10} = &\frac{3}{16}\cos\theta(r^3 - r) \\ &+ \frac{\mathrm{Re}^2\cos\theta}{9216}\left[\frac{r^9}{80} - \frac{r^7}{8} + \frac{3r^5}{8} - \frac{r^3}{2} + \frac{19r}{80}\right], \end{aligned} \quad \ldots (30)$$

$$\begin{aligned} w_{11} = &\Bigg[\frac{1}{i\alpha^2}\frac{1}{J_1(i^{3/2}\alpha)}\left(\frac{A}{i\alpha^2} - p_1\right) + C + E \\ &+ (P + R + T + V)\frac{J_0(i^{3/2}\alpha)}{J_1(i^{3/2}\alpha)}\Bigg]J_1(i^{3/2}\alpha r) \\ &+ ((Cr^6 + Er^4)J_1(i^{3/2}\alpha r) + \frac{\mathrm{Re}\, r\cos\theta}{i\alpha^2}\left(\frac{A}{i\alpha^2} - p_1\right) \\ &+ (\mathrm{P}r^7 + Rr^5 + Tr^3 + Vr)J_0(i^{3/2}\alpha r))\cos\theta \end{aligned} \quad \ldots (31)$$

where,

$$S = i\alpha^2\Bigg[\begin{aligned} &\frac{2J_1(-i^{3/2}\alpha)\sin\theta}{i^{3/2}\alpha\left(J_2(i^{3/2}\alpha) - J_0(i^{3/2}\alpha)\right) + 2J_1(i^{3/2}\alpha)} \\ &\Bigg[-\frac{\mathrm{Re}^2 p_1}{4\alpha^4} + \frac{\mathrm{Re}^2 p_1}{i\alpha^2}\left(\frac{1}{96} + \frac{i}{15\alpha^2}\right) \\ &+ \frac{\mathrm{Re}^2 p_1}{i\alpha^2}\frac{J_2(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\left(\frac{-7}{480} - \frac{1}{15i\alpha^2}\right) \\ &+ \frac{\mathrm{Re}^2 p_1}{i^{1/2}\alpha}\frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\left(\frac{-2i}{5\alpha^4} - \frac{i}{240} - \frac{1}{16\alpha^2}\right)\Bigg] \\ &- \frac{\mathrm{Re}^2 p_1}{8\alpha^4} - \frac{\mathrm{Re}^2 p_1}{i\alpha^6} - \frac{\mathrm{Re}^2 p_1}{i\alpha^2}\Bigg(\frac{1}{15i\alpha^2} \\ &- \frac{1}{240}\left(\frac{7i^{1/2}}{240\alpha} + \frac{2}{15i^{3/2}\alpha^3}\right)\frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\Bigg) \end{aligned}\Bigg],$$

$$P = \frac{\mathrm{Re}^2 p_1}{i4608 \times 14\alpha^2 J_0(i^{3/2}\alpha)},$$

$$C = \frac{47\mathrm{Re}^2 p_1}{12\alpha^3 i^{1/2} \times 6720 J_0(i^{3/2}\alpha)},$$

$$V = \left(\frac{-p_1}{2\alpha^2 J_0(i^{3/2}\alpha)} - \frac{B}{2i^{3/2}\alpha} - \frac{2\mathrm{Re}^2 p_1}{4608 i\alpha^2 J_0(i^{3/2}\alpha)}\right),$$

$$R = \frac{\mathrm{Re}^2 p_1}{10\alpha^2 J_0(i^{3/2}\alpha)}\left(\frac{i}{768\alpha} - \frac{1}{20i\alpha^3} - \frac{47}{3360\alpha^3}\right),$$

$$\begin{aligned} E = &\frac{\mathrm{Re}^2 p_1}{8i^{3/2}\alpha J_0(i^{3/2}\alpha)}\Bigg(\frac{1}{15\alpha^4} + \frac{1}{96i\alpha^2} + \frac{24}{7680i\alpha^2} \\ &- \frac{12i}{100\alpha^4} + \frac{1}{1400\alpha^4}\Bigg) \end{aligned}$$

$$B=\frac{2}{i^{3/2}\alpha\left(J_2(i^{3/2}\alpha)-J_0(i^{3/2}\alpha)\right)+2J_1(i^{3/2}\alpha)}\left[-\frac{\mathrm{Re}^2 p_1}{4\alpha^4}\right.$$
$$+\frac{\mathrm{Re}^2 p_1}{i\alpha^2}\left(\frac{1}{96}+\frac{i}{15\alpha^2}\right)$$
$$+\frac{\mathrm{Re}^2 p_1}{i\alpha^2}\frac{J_2(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\left(\frac{-7}{480}-\frac{1}{15i\alpha^2}\right)$$
$$\left.+\frac{\mathrm{Re}^2 p_1}{i^{1/2}\alpha}\frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\left(\frac{-2i}{5\alpha^4}-\frac{i}{240}-\frac{1}{16\alpha^2}\right)\right]$$

$$T=\frac{\mathrm{Re}^2 p_1 i^{-1/2}}{6\alpha J_0(i^{3/2}\alpha)}\left(\frac{9}{4608i^{1/2}\alpha}+\frac{2i}{15i^{1/2}\alpha^5}+\frac{1}{48i^{1/2}\alpha^3}-\frac{1}{15i^{1/2}\alpha^5}\right.$$
$$\left.-\frac{1}{96i^{3/2}\alpha^3}-\frac{24}{7680i^{3/2}\alpha^3}-\frac{12}{100\alpha^2}+\frac{1}{1400i^{1/2}\alpha^5}\right)$$

$$A=BJ_1(i^{3/2}\alpha)+\left(\frac{-\mathrm{Re}^2 p_1}{8\alpha^4}-\frac{p_1}{i\alpha^6}-\frac{p_1}{i\alpha^2}\left(\frac{1}{15i\alpha^2}\right.\right.$$
$$\left.\left.-\frac{1}{240}\left(\frac{7i^{1/2}}{240\alpha}+\frac{2}{15i^{3/2}\alpha^3}\right)\frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\right)\right).$$

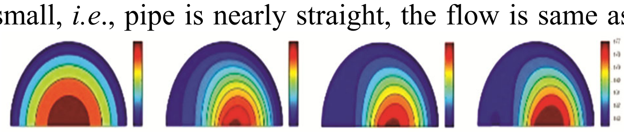
As the radius of curvature is large when compare to the radius of cross section of the pipe, the curvature ratio (δ) is less than 1 and hence its higher powers will be much lesser than 1. In view of this, we have solved the associated boundary value problems up to order 1. Here, only real part of the solution is considered while plotting the contours as it represent the solution of the underlying problem.

One can observe that the solution contains terms involving powers of Re^2 as well and hence it may not be convergent for arbitrary values of Re. Therefore, to guarantee the convergence of solution, we take $\mathrm{Re}^2\delta \le 4608$.

4 Results and discussion

In this section, the authors present the effect of Reynolds number (Re), curvature ratio (δ), Womersley number (α) and time (t) on the axial velocity of the fluid.

In Fig. 2, when the curvature ratio is very very small, *i.e.*, pipe is nearly straight, the flow is same as that of the normal Poiseuille flow; see Fig. 2(a). As bend of the pipe increases, i.e., as δ increases, which means the curvature of the pipe increase, the secondary flow sets-in and it superimpose on the main stream flow; as a result, there is a shift of axial velocity towards the outer boundary of the pipe to balance the centrifugal force arising due to the curvature of pipe; see Fig. 2(b-c). However, due to the nature of centrifugal forces, the maximum axial velocity occurs near to the outer boundary of the pipe (not towards inner boundary of pipe). If the curvature ratio increases further, secondary flow significantly affects the flow pattern and another vortex starts forming in opposite direction; see Fig. 2 (d).

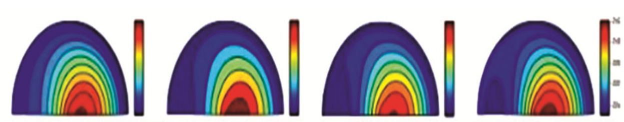


(a) δ=0.0001 (b) δ=0.01 (c) δ=0.05 (d)δ=0.1

Fig. 2 — Effect of δ on axial velocity when Re=200, t=0 and α=1.

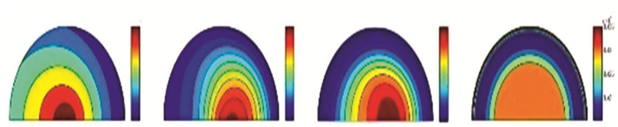
Figure 3 presents the effect of Reynolds number on the axial velocity contours. The intensity of centrifugal force which was set-in increase with the increase in Reynolds number Re; hence axial velocity contours affected accordingly and shifts towards the outer boundary of pipe; see Fig. 3(a-c). Generally, increase of Reynolds number corresponds to high velocity or lesser viscosity. In both cases, even if the curvature is fixed, there is a rise in the centrifugal force, due to which the axial velocity contours are affected accordingly. After certain value of Reynolds number, a second contour starts to form in the inner region, in the opposite direction; see Fig. 3(d). This is an indication of the secondary flow. As Re increases, the axial velocity of the flow also increases.

Figure 4 shows the effect of Womersley number on fluid flow. As Womersley number is a dimensionless expression of the of pulsatile flow frequency in relation to viscous effects, we observe shift in the velocity profile when Womersley number increases (see Fig. 4(a-c)) to balance the centrifugal force. One



(a) Re=200 (b) Re=400 (c) Re=650 (d) Re=700

Fig. 3 — Effect of Re on axial velocity when δ=0.01, t=0 and α=1



(a) α=0.01 (b) α=1 (c) α=5 (d) α=10

Fig. 4 — Effect of α on axial velocity when Re=200, δ=0.01 and t=0

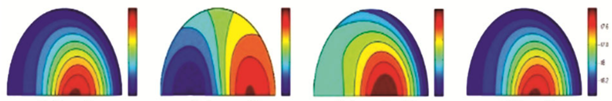


Fig. 5 — Effect of t on axial velocity when Re=200, δ=0.01 and α=1

can note from Fig. 4(d) that there is a drastic change in the axial velocity contour when α=10. This behavior is qualitatively similar to that of earlier studies (Sigger and Waters[18]).

From Fig. 5, it is observed that the velocity pattern is qualitatively periodic in nature, which is expected as the flow is due to the pulsatile pressure gradient.

5 Conclusions

The research reported in this paper is considering the pulsatile Newtonian fluid flow through a curved pipe of circular cross section. Coupled partial differential equations governing the flow are solved analytically using perturbation series. Following are the outcomes of the current study:

- It has been observed that, for very small curvature ratio, the axial velocity is qualitatively same as that of unsteady pulsatile flow through straight pipe. As curvature ratio increases, to balance the centrifugal force, axial velocity shifts towards the outer boundary. If curvature is further increased, the secondary flow starts to set-in and hence, we observe another vortex in velocity contour in opposite direction.
- Increase in Reynolds number highly affects the axial velocity similar to that of highly curved pipes.
- Womersley number, if small, has lesser impact on the velocity profile, as lesser Womersley number corresponds to the lesser pulsatile pressure. But higher Womersley number (α>10) highly affects the axial velocity; in this case, axial velocity in the middle region is high compared to the axial velocity near the boundary of pipe.
- The flow is qualitatively periodic with respect to time.

Nomenclature

a	Radius of pipe
R	Radius of curvature of pipe
Re	Reynolds number
t	Time
u,v,w	Components of fluid velocity
α	Womersley number
δ	Curvature Ratio
k	Curvature of pipe
μ	Viscosity of fluid
ϱ	Density of fluid
ψ	Stream function
ω	Angular frequency

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