Three dimensional viscous flow and heat transfer due to a permeable shrinking sheet with heat generation/absorption

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Three dimensional boundary layer flow due to a permeable shrinking sheet with viscous dissipation and heat generation/absorption, has been studied in the present paper. The governing equations are transformed to ordinary differential equations by using suitable similarity transformations and then solved numerically on computer by standard technique. Numerical results of velocity and temperature profiles are obtained with the effects of various parameters involved such as suction, shrinking, Prandtl number, Eckert number and heat generation coefficient etc. and discussed them graphically in suitable manner such that interesting aspects of the solution can be adopted. Also, the comparison of results of two dimensional case and axisymmetric shrinking sheet case is considered.

Keywords: Viscous flow, Shrinking sheet, Suction, Viscous dissipation, Heat generation/absorption

1 Introduction

The viscous flow of an incompressible fluid over a permeable shrinking/stretching sheet has many applications in manufacturing industries and technological process, such as, glass-fiber production, wire drawing, paper production, metal and polymer processing industries and many others. The boundary layer flow over a stretching sheet was first studied by Sakiadis[1]. Later, Crane[7] found a closed form solution for steady two dimensional flows over a stretching sheet where the velocity on the boundary is away and proportional to the distance. The study of heat and mass transfer over a stretching sheet subject to suction or blowing (injection) was investigated by Gupta and Gupta[10]. Whereas Wang[2] studied the three dimensional flow due to a stretching surface. Additively, flow of visco-elastic fluid over a stretching sheet was investigated by Rajagopal and Gupta[5]. Bhattacharya and Gupta[15] extended the idea on the stability of viscous flow over a stretching sheet. Uniqueness of flow of a second order fluid past a stretching sheet was investigated by Troy *et al*[17]., Vajravelu and Hadjinicalaou[6] worked out on heat transfer in a viscous fluid over a stretching sheet with viscous dissipation and internal heat generation. Whereas, the axis symmetrically motion of a liquid film on an unsteady stretching surface was studied by Usha and Sridharan[11]. Ariel[9] gave generalized three dimensional flows due to a stretching sheet. Further, three dimensional flow over a stretching surface with

thermal radiation and heat generation in the presence of the chemical reaction and suction/injunction was considered by Elbashbeshy *et al*[4].

Recently, the boundary layer flow due to a shrinking sheet has attracted more considerable interest. Crane's[7] stretching sheet solution induces a far field suction towards the sheet, while flow over a shrinking sheet would give rise to a velocity away from the sheet. From a physical point of view, vorticity generated at the shrinking sheet is not confined within a boundary layer and a steady flow is not possible unless adequate suction is applied at the surface. For this type of shrinking flow, it is essentially a backward flow as discussed by Goldstein[12]. For a backward flow configuration, the fluid losses any memory of the perturbation introduced by the sheet. As a result, the flow induced by the shrinking sheet shows quite distinct physical phenomena from the forward stretching case. Miklavcic and Wang[8] investigated two-dimensional and axisymmetric viscous flow induced by a shrinking sheet in the presence of uniform suction. The above shrinking sheet problem was extended to power-law surface velocity by Fang[16]. While, Bhattacharyya[13] reported boundary layer flow and heat transfer over an exponentially shrinking sheet. Stagnation flow towards a shrinking sheet was given by Wang[3]. Further, effects of suction/blowing on steady boundary layer stagnation flow and heat transfer towards a shrinking sheet with thermal

radiation was investigated by Bhattacharyya and Layek[14].

The object of present paper is to study the effects of viscous dissipation and heat generation/absorption on three dimensional viscous flow due to a permeable axisymmetric shrinking sheet and then compare the results of above problem with the problem of the two dimensional flow due to a shrinking sheet.

2 Formulation of the Problem

Consider a three-dimensional viscous flow of an incompressible fluid due to a permeable shrinking sheet in the presence of heat generation or absorption. The sheet coincides with the plane $z=0$ and the flow is confined in the region $z>0$. The x and y axes are taken along the length and breadth of the sheet and z-axes is perpendicular to the sheet, respectively (Fig. 1). A suction is applied normal to sheet to contain the vorticity. Let (u, v, w) be the velocity components along the (x, y, z) directions, respectively. Under the usual boundary layer approximations, the basic governing boundary layer equations (Miklavcic and Wang[8] and Wang[2]) with viscous dissipation are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \ldots (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\left(\frac{\partial^2 u}{\partial z^2}\right) \qquad \ldots(2)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+v\left(\frac{\partial^2 v}{\partial z^2}\right) \qquad \ldots(3)$$
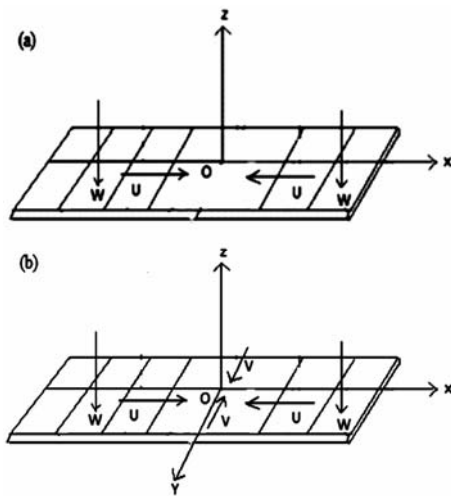

Fig. 1 — Physical model and coordinate system for (a) two dimensional shrinking sheet case and (b) axisymmetric shrinking sheet case

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+v\left(\frac{\partial^2 w}{\partial z^2}\right) \qquad \ldots(4)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{\kappa}{\rho c_p}\left(\frac{\partial^2 T}{\partial z^2}\right)$$
$$+\frac{\mu}{\rho c_p}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]+\frac{Q(T-T_\infty)}{\rho c_p} \qquad \ldots(5)$$

where p is the pressure, ρ the density, μ the dynamic viscosity, $v=\dfrac{\mu}{\rho}$ the kinematic viscosity, κ thermal conductivity of the fluid, C_p the specific heat at constant pressure and Q is the volumetric rate of heat generation or absorption.

The boundary conditions applicable to the present flow are:

$$Z = 0: u=-U=-ax,\ v=-V=-a(m-1)y,\ w=-W,$$
$$\qquad T=T_w$$
$$z\to\infty: u\to 0,\ v\to 0,\ T\to T_\infty \qquad \ldots (6)$$

where $a>0$ is the shrinking constant, U and V the shrinking velocities, $W>0$ the suction velocity, m shrinking parameter, T_w the variable sheet temperature and T_∞ is the stream temperature. Also, far from the sheet the pressure is uniform.

3 Analysis

Introducing the following similarity transformations:

$$u=axf'(\eta),\ v=a(m-1)yf'(\eta),\ w=-\sqrt{av}mf(\eta),$$
$$\eta=z\sqrt{\frac{a}{v}}\ \text{and}\ \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty} \qquad \ldots (7)$$

Eq. (1) is identically satisfied by similarity transformations given in Eq. (7) while Eq. (4) can be integrated to give:

$$\frac{p}{\rho}=v\frac{\partial w}{\partial z}-\frac{w^2}{2}+\text{Constant} \qquad \ldots (8)$$

Further the problem can be classified by considering different values of the shrinking parameter m when $m=1$ the sheet shrinks in one direction only and when $m=2$, the sheet shrinks axisymmetrically etc.

Case (a):

When $m=1$ [i.e. sheet shrinks in one direction only i.e. x-direction (say)] gives $v=0$ i.e. system becomes two dimensional and also, it is assumed that the temperature difference between the sheet and the stream varies as:

$$T_w - T_\infty = Ax \qquad \qquad \ldots (9)$$

where A is a constant and x is the distance measured from the leading edge of the surface.

Then, with the help of Eqs (7) and (9), the governing Eqs (2) and (5) become:

$$f''' + ff'' - f'^2 = 0 \qquad \qquad \ldots(10)$$

$$\theta'' + \Pr\left[f\theta' - f'\theta + B\theta + Ecf''^2\right] = 0 \qquad \ldots(11)$$

The corresponding boundary conditions are:

$$\eta = 0: \ f = S, \ f' = -1, \ \theta = 1$$
$$\eta \to \infty: \ f' \to 0, \ \theta \to 0 \qquad \qquad \ldots(12)$$

where a prime denotes differentiation with respect to similarity parameter η, $S = \dfrac{W}{\sqrt{av}}$ is the Suction parameter, $\Pr = \dfrac{\mu c_p}{\kappa}$ is the Prandtl number, $B = \dfrac{Q}{a\rho c_p}$ is the Heat Source ($B>0$) or Sink ($B<0$) parameter and $Ec = \dfrac{U^2}{c_p(T_w - T_\infty)}$ is the Eckert number.

The physical quantities of interest are the local skin friction coefficient C_f on the surface and the Nusselt number Nu i.e. surface heat transfer are given by:

$$C_f = \frac{\tau_w}{\rho U^2/2} = \frac{\mu\left(\dfrac{\partial u}{\partial z}\right)_{z=0}}{\rho U^2/2} = \frac{2}{\sqrt{\text{Re}}}f''(0)$$

$$Nu = -\frac{x\left(\dfrac{\partial T}{\partial z}\right)_{z=0}}{(T_w - T_\infty)} = -\sqrt{\text{Re}}\,\theta'(0) \qquad \ldots(13)$$

where, $\text{Re} = \dfrac{Ux}{v}$ is the Reynolds number.

Now, the set of nonlinear ordinary differential Eqs (10) and (11) with boundary condition given in Eq.(12) are solved numerically using Runge-Kutta forth order algorithm with a systematic guessing of $f''(0)$ and $\theta'(0)$ by the shooting technique until the boundary conditions at infinity satisfied. The step size $\eta=0.003$ is used while obtaining the numerical solution and accuracy up to the seventh decimal place i.e. 1×10^{-7} which is very sufficient for convergence. In this method, we choose suitable finite values of $\eta\to\infty$, say η_∞, which depend on the values of the parameters used. Also, all the computations are done by a program which uses a symbolic and computational computer language Matlab.

The flow due to shrinking sheet was studied by Miklavcic and Wang[8]. According to their analyses, the steady two-dimensional flow due to a shrinking sheet with suction parameter S will appear only when S is greater than or equal to 2. However, from our investigation it is found that if the suction parameter is greater than or equal to 1.85 (which is near to 2), then only the steady flow due to a shrinking sheet is possible, which shows a favourable agreement and give confidence that the numerical results obtained are accurate. Hence, the similarity solution exists when the suction parameter S satisfies the condition $S\geq1.85$ and consequently for $S<1.85$ the flow has no similarity solution. In this regard, the values of the skin friction coefficient $f''(0)$ for different values of S are shown in Fig. 2. The values of temperature gradient at the sheet $-\theta'(0)$ which are proportional to the rate of heat transfer from the sheet are plotted in Fig. 3 against S for different values of the Prandtl number Pr heat source/sink parameter B and Eckert number Ec. Further, it is also observed from Fig. 2 that the skin friction coefficient $f''(0)$ increases with S. While, Fig. 3 dissipates that the value of $-\theta'(0)$ increases with increasing S and Pr and decreases with increasing B and Ec.
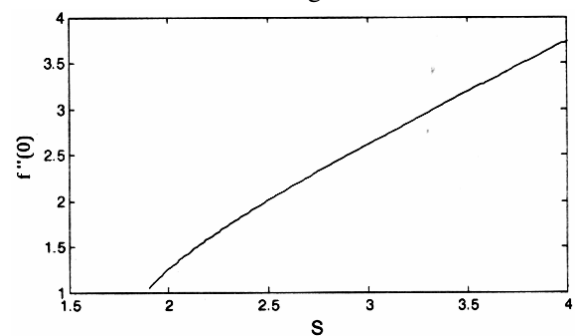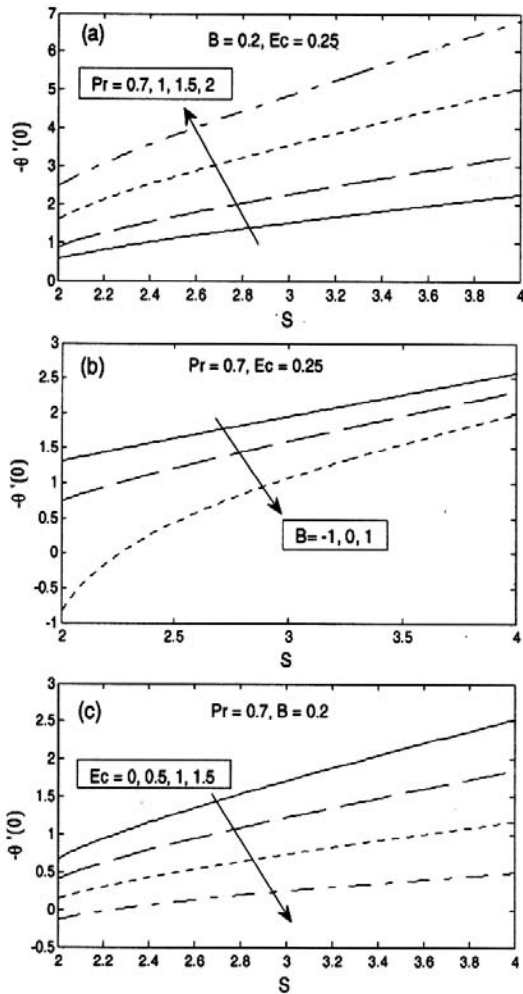

Fig. 2 — Skin-friction coefficient against suction parameter

Fig. 4 — Velocity profile against η for various values of suction parameter S

Fig. 5 — Temperature profile against η for various values of suction parameter S

Fig. 6 — Temperature profile against η for various values of Prandtl number Pr

Fig. 7 — Temperature profile against η for various values of heat source/sink parameter B

Fig. 3 — Rate of heat transfer at sheet against suction parameter for various values of (a) Prandtl number, (b) heat source/sink parameter and (c) Eckert number

The effects of suction parameter, Prandtl number, heat source/sink and Eckert number on the velocity and temperature profiles are shown in Figs 4-8. It is observed from the Figs 4 and 5 that the width of the velocity boundary layer and thermal boundary layer both decrease for increasing values of the parameter S.

Figure 6 shows Prandtl number effects on temperature profile, it is predicted the temperature of fluid as well as thermal boundary layer thickness decreases with increase of Prandtl number. This happens because when Pr increases, the thermal conductivity decreases, thus it leads to the decrease of the energy transfer ability. The effects of heat source/sink B on the temperature distribution dissipates in Fig. 7 and it is found that the temperature of fluid decreases for the increasing strength of heat
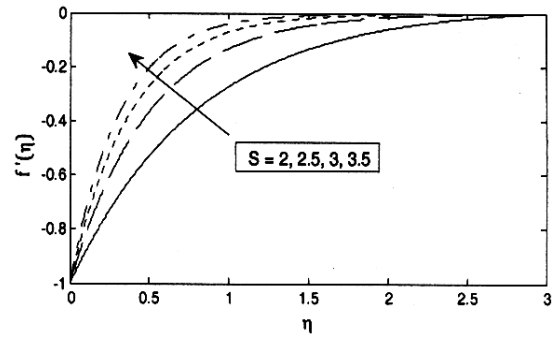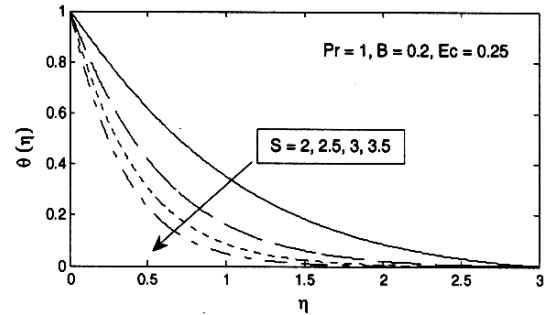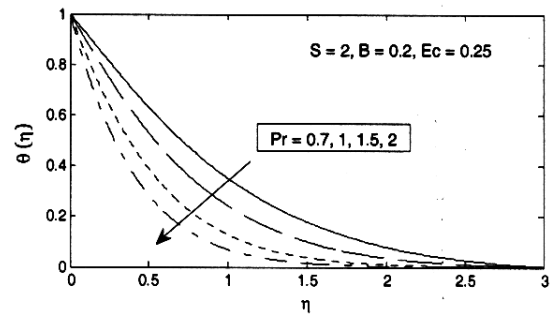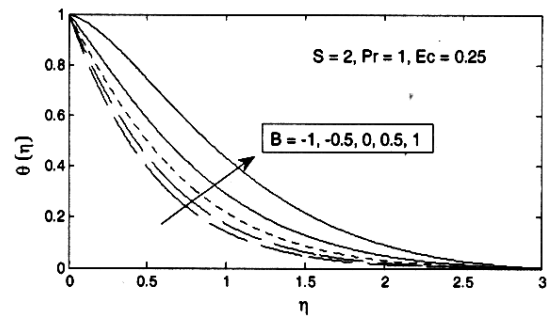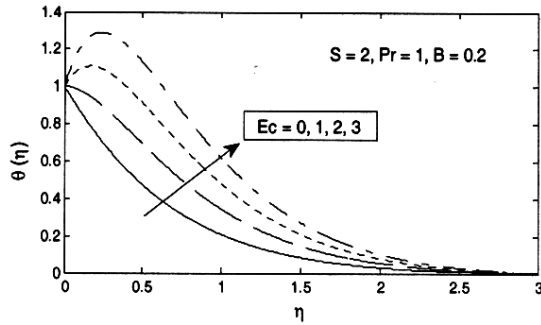
Fig. 8 — Temperature profile against η for various values of Eckert number Ec

sink $B<0$ while due to increase of heat source strength $B>0$ the temperature increases. Therefore, thickness of the thermal boundary layer reduces for increasing of the heat sink but it increases with heat source parameter. From Fig. 8, it is noticed that by increasing values of Ec the temperature profile increases near the sheet. This is due to the fact that heat energy is stored in the fluid due to the frictional heating. The strong frictional heating slows down the cooling process of the sheet and in this case the study suggests that the rapid cooling of the sheet can be made possible if the viscous dissipation can be made as small as possible.

Case (b):

When $m=2$ (i.e. sheet shrinks axisymmetrically), the flow system becomes three dimensional and also, it is assumed that the temperature difference between the sheet and the stream varies as (Elbashbeshy et al[4].)

$$T_w - T_\infty = A_0 x = A_1 y \qquad \ldots(14)$$

where A_0 and A_1 are constants.

Therefore, with the help of Eqs (7 and 14), the governing Eqs (2) and (3) reduce to same Eq. (15) and Eq. (5) reduces to Eq. (16) as:

$$f''' + 2ff'' - f'^2 = 0 \qquad \ldots(15)$$

$$\theta'' + \Pr\left[2f\theta' - 2f'\theta + B\theta + \{Ec_x + Ec_y\}f''^2\right] = 0 \qquad \ldots(16)$$

The corresponding boundary conditions are:

$$\eta = 0: \quad f = S, \quad f' = -1, \quad \theta = 1$$
$$\eta \to \infty: \quad f' \to 0, \quad \theta \to 0 \qquad \ldots(17)$$

where $S = \dfrac{W}{2\sqrt{av}}$ is the Suction parameter,

$$Ec_x = \frac{U^2}{c_p(T_w - T_\infty)} = \frac{a^2 x}{c_p A_0} \text{ and}$$

$$Ec_y = \frac{V^2}{c_p(T_w - T_\infty)} = \frac{a^2 y}{c_p A_1} \text{ are the local Eckert}$$

numbers based on the x and y variables, respectively.

The physical quantities of interest are the local skin friction coefficient C_f on the surface along the x and y directions, which are denoted by C_{fx} and C_{fy}, respectively and the Nusselt number Nu i.e. surface heat transfer are given by:

$$C_{fx} = \frac{\tau_{wx}}{\rho U^2/2} = \frac{\mu\left(\dfrac{\partial u}{\partial z}\right)_{z=0}}{\rho U^2/2} = \frac{2}{\sqrt{\mathrm{Re}_x}} f''(0)$$

$$C_{fy} = \frac{\tau_{wy}}{\rho V^2/2} = \frac{\mu\left(\dfrac{\partial v}{\partial z}\right)_{z=0}}{\rho V^2/2} = \frac{2}{\sqrt{\mathrm{Re}_y}} f''(0)$$

$$Nu = -\frac{x\left(\dfrac{\partial T}{\partial z}\right)_{z=0}}{(T_w - T_\infty)} = -\sqrt{\mathrm{Re}_x}\,\theta'(0) \qquad \ldots(18)$$

where τ_{wx} and τ_{wy} are the wall shear stresses along the x and y directions, respectively and $\mathrm{Re}_x = \dfrac{ax^2}{v}$ and $\mathrm{Re}_y = \dfrac{ay^2}{v}$ are the Reynolds numbers based on x and y variables, respectively.

The set of nonlinear ordinary differential Eqs (15) and (16) with boundary condition given in Eq. (17) are also solved numerically using Runge - Kutta forth order algorithm with the shooting technique as of in case (a). Now, according to Miklavcic and Wang[8] analyses of the steady three-dimensional viscous flow due to an axisymmetric shrinking sheet with suction parameter S, there exist infinitely many solutions near $S = \sqrt{2}$. However, from our investigation, it is found that if the suction parameter is greater than or equal to 1.25 (which is near to $\sqrt{2}$) then the steady flow is possible, which shows a favourable agreement and give confidence that the numerical results obtained are accurate. Hence, the similarity solution exists when the suction parameter S satisfies the

condition $S \geq 1.25$ and consequently for $S < 1.25$ the flow has no similarity solution due to a sheet which shrinks axisymmetrically in its own plane. In this regard, the values of the skin friction coefficient $f''(0)$ for different values of S are shown in Fig. 9. The values of temperature gradient at the sheet $-\theta'(0)$ which are proportional to the rate of heat transfer from the sheet are plotted in Fig. 10 against S for different values of the Prandtl number Pr, heat source/sink parameter B and local Eckert numbers Ec_x and Ec_y. Further, it is also observed from Fig. 9 that the skin friction coefficient $f''(0)$ increases with S. While, Fig. 10 shows that the value of $-\theta'(0)$ increases with increasing S and Pr and decreases with increasing B, Ec_x and Ec_y.

Consequently, the effects of suction parameter on the velocity and temperature profiles are shown in Figs 11 and 12, respectively. It is observed from Figs 11 and 12 that the velocity boundary layer and thermal boundary layer both decrease for increasing values of the parameter S. Figure 13 shows that the thermal boundary layer thickness decreases with increase of Prandtl number.

The effects of heat source/sink B on the temperature distribution dissipates in Fig. 14 and it is found that the thickness of the thermal boundary layer reduces for increasing of the heat sink but it increases with heat source parameter. From Figs 15 and 16, it is noticed that by increasing values of Ec_x and Ec_y the temperature profile increases near the wall due to the fact that heat energy is stored in the fluid due to the frictional heating. Thus, we observed that the effects of suction parameter, Prandtl number, heat source/sink and Eckert numbers for the two dimensional case and axisymmetric case on the velocity profile and temperature profile are likely the same. Figs 17 and 18 show the comparison of velocity profile and temperature profile for these two cases. It is observed from both Figs 17 and 18 that the velocity
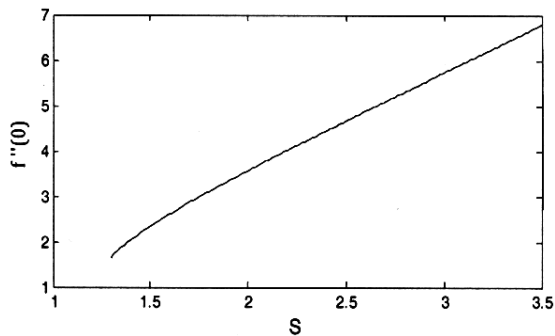
of the fluid is greater while the temperature of fluid is less for axisymmetric sheet instead of sheet shrinks in one direction only i.e. two dimensional case. In other words we can say that in the case of axisymmetric shrinking sheet the widths of velocity boundary layer as well as that of the thermal boundary layer both are less than those of the case of two dimensional.
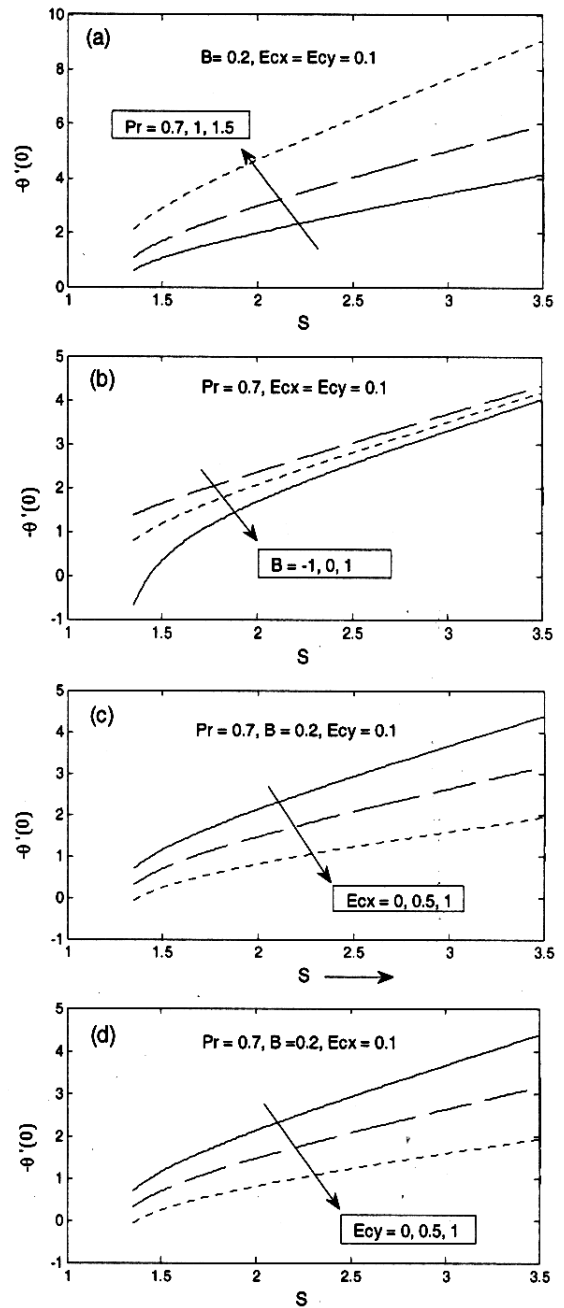

Fig. 10 — Rate of heat transfer at sheet against suction parameter for various values of (a) Prandtl number, (b) heat source/sink parameter, (c) local Eckert number Ec_x and (d) local Eckert number Ec_y
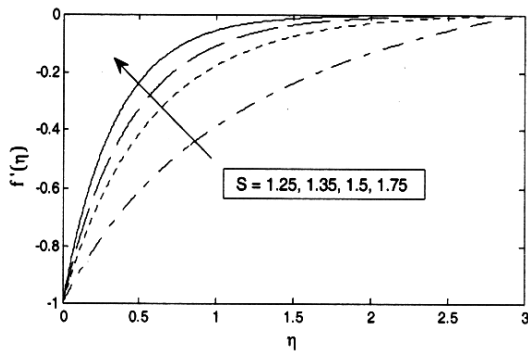

Fig. 9 — Skin-friction coefficient against suction parameter

Fig. 11 — Velocity profile against η for various values of suction parameter S

Fig. 12 — Temperature profile against η for various values of suction parameter S
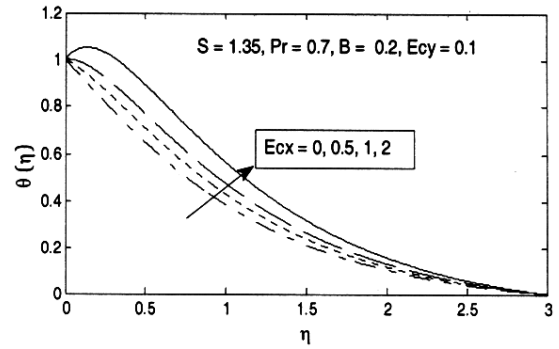

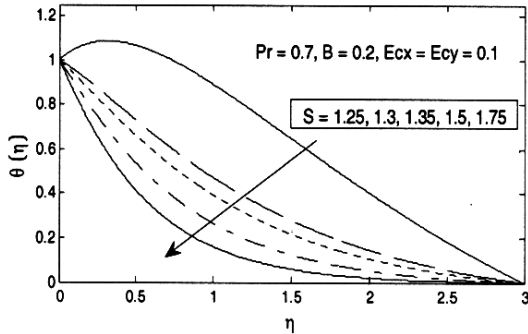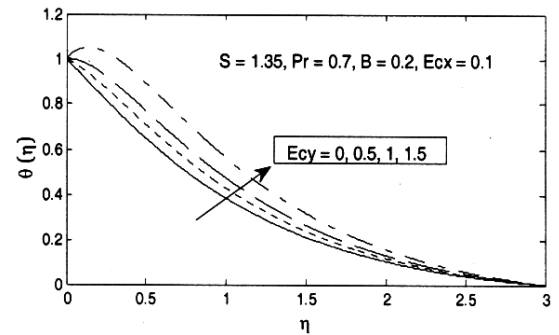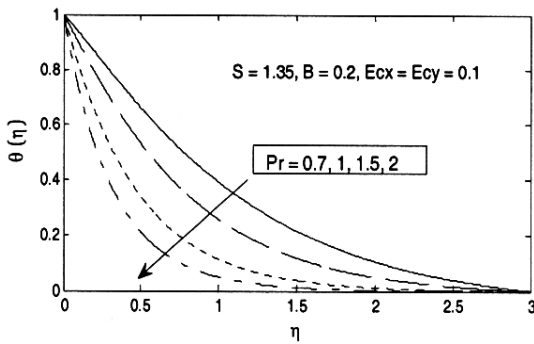Fig. 13 — Temperature profile against η for various values of Prandtl number Pr

Fig. 14 — Temperature profile against η for various values of heat source/sink parameter B

Fig. 15 — Temperature profile against η for various values of local Eckert number Ec_x

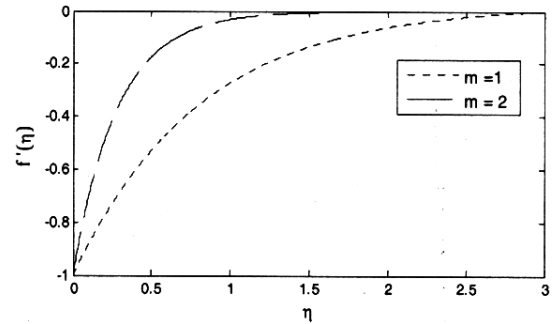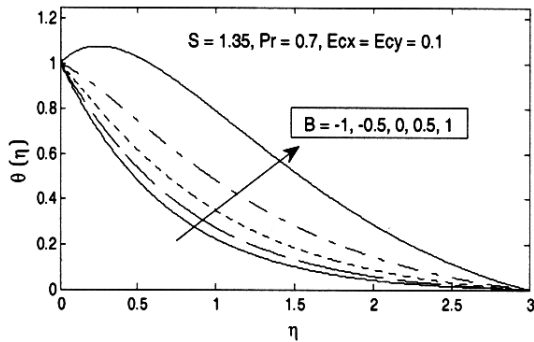Fig. 16 — Profile against η for various values of local Eckert number Ec_y

Fig. 17 — Velocity profile against η for various values of shrinking parameter m
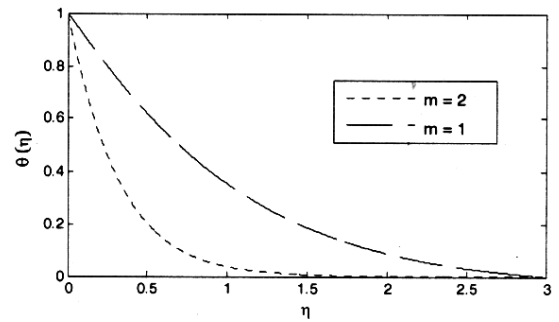

Fig. 18 — Temperature profile against η for various values of shrinking parameter m

Table 1 — Values of $f''(0)$ and $-\theta'(0)$ with S and m

	m	S		
		2	2.25	2.5
$f''(0)$	1	1.25602	1.66566	2.00829
	2	3.57468	4.13342	4.67673
$-\theta'(0)$	1	0.57297	0.86218	1.09898
	2	1.99634	2.38549	2.75414

Table 1 presents the skin-friction coefficient and rate of heat transfer at sheet for different values of suction parameter for both cases and it is observed that for each value of suction parameter the skin-friction coefficient and rate of heat transfer at sheet both are higher for the case of axisymmetric sheet than two dimensional case.

4 Conclusions

The effects of viscous dissipation and heat generation/absorption on two dimensional viscous flow past a shrinking sheet and three dimensional viscous flow past an axisymmetric shrinking sheet with suction are investigated.

It is observed that velocity profile increases with suction parameter and the tempreture profile increases for increasing values of heat generation cofficient as well as Eckert numbers but decreases with suction parameter and Prandtl number for both cases when sheet shrinks in one direction only and when the sheet shrinks axisymmetrically. The effect of viscous dissipation is to increase the temperature near the surface of the shrinking sheet and therefore to make the cooling process efficient it is strongly recommended that the viscous dissipation must be made as small as possible.

In both cases the skin friction coefficient increases with suction parameter and the rate of heat transfer at surface of the sheet increases with suction parameter and Prandtl number while decreases with heat source/sink parameter and Eckert numbers. The

difference in both cases we find that the velocity boundary layer and thermal boundary layer both are high in two dimensional case instead of axisymmetric sheet case, whereas the skin-friction coefficient and rate of heat transfer at sheet both are higher for the case of axisymmetric sheet than two dimensional case.

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